ASYMPTOTIC EXPANSIONS FOR THE MEAN SQUARE OF HIGHER DERIVATIVES OF LERCH ZETA-FUNCTIONS

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Let $s=\sigma+it$ be a complex variable, x and λ real parameters with $x>0$. The Lerch zeta-function $\phi(s,x,\lambda)$ is defined by the Dirichlet series $\sum_{l=0}^{\infty}e^{2\pi il\lambda}(l+x)^{-s}$ ($\operatorname{Re}s=\sigma>1$) and its meromorphic continuation over the whole s-plane; this reduces to the Hurwitz zeta-function when $\lambda\in\mathbb{Z}$, and further to the Riemann zeta-function if $x=1$. The initial object of this talk is a multiple mean square of $\phi(s,x,\lambda)$ in the form

$$I_m(s;a,\lambda)=\int_0^1\cdots\int_0^1|\phi(s,a+x_1+\cdots+x_m,\lambda)|^2dx_1\cdots dx_m, \tag{1}$$

where $m\geq 1$ is any integer and $a>0$ any real number. The particular case $m=1$ and $a=1$ of (1) was first treated by Zhang [Zh2], who obtained its asymptotic formula as $\operatorname{Im}s=t\to+\infty$ for any $\lambda\in\mathbb{R}$, with the error term $O(t^{-1})$. A complete asymptotic expansion of $I_1(s;1,\lambda)$ in the descending order of t as $t\to\pm\infty$ was established by the author [Ka1], who further showed recently in [Ka2] that a similar asymptotic series still exists for (1), even in the most general setting. The initial aim of this talk is to present the complete asymptotic expansion of (1) above in the descending order of t as $t\to\pm\infty$.

Next let $\phi^{(m)}(s,x,\lambda)=(\partial/\partial s)^m\phi(s,x,\lambda)$ for $m=0,1,\dots,$ and define

$$J^{(m_1,m_2)}(s_1,s_2;a,\lambda)=\int_0^1\phi^{(m_1)}(s_1,a+x,\lambda)\phi^{(m_2)}(s_2,a+x,-\lambda)dx, \tag{2}$$

where $m_1\geq 0$ and $m_2\geq 0$ are arbitrarily fixed integers. The particular case $m_1=m_2=1$, $a=1$ and $\lambda\in\mathbb{Z}$ of (2) was first studied by Zhang [Zh1], who derived its asymptotic formula as $t\to+\infty$ on the critical line $s_1=1/2+it$ and $s_2=1/2-it$, with the error term $O(t^{-1/6}\log^{10/3}t)$. Our next topic is then to show that a complete asymptotic expansion of (2) as $t\to\pm\infty$ further exists if $s_1=\sigma_1+it$ and $s_2=\sigma_2-it$ with any real σ_1 and σ_2, even in the most general setting; this extends and refines upon our previous result [KM, Theorem 3], which asserts that an asymptotic formula as $t\to+\infty$ holds for (2) on the critical line $\sigma_1=\sigma_2=1/2$, with the error term $O(t^{-m-1})$, in particular when $m_1=m_2=m$, $a=1$ and $\lambda\in\mathbb{Z}$.

References

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