Functional relations for zeta-functions of root systems

Yasushi Komori

Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan
Present address: Department of Mathematics, Rikkyo University
Nishi-Ikebukuro, Toshima-ku, Tokyo 171-8501, Japan
E-mail: komori@rikkyo.ac.jp

Kohji Matsumoto

Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan
E-mail: kohjimat@math.nagoya-u.ac.jp

Hirofumi Tsumura

Department of Mathematics and Information Sciences, Tokyo Metropolitan University
1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397 Japan.
E-mail: tsumura@tmu.ac.jp

We report on the theory of functional relations among zeta-functions of root systems, including known formulas for their special values. In the first part of this paper, we present known results on value-relations and functional relations for zeta-functions of root systems of A_2 type. Also, in view of the symmetry of underlying Weyl groups, we discuss a general framework of functional relations. In the second part of this paper, we prove several new results; we give a method for constructing functional relations systematically, and prove new functional relations among zeta-functions of root systems of types A_3, $C_2(\simeq B_2)$, B_3 and C_3, which include Witten's volume formulas as value-relations with explicit values of coefficients.

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1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0$ the set of non-negative integers, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

The multiple zeta value (MZV) of depth r is defined by

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{m_1 > m_2 > \cdots > m_r \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \tag{1.1}$$

for $k_1, k_2, \ldots, k_r \in \mathbb{N}$ with $k_1 > 1$ (see Zagier [51] and Hoffman [12]). It was Euler who first studied the double zeta values and gave some relation formulas among them such as

$$\sum_{j=2}^{k-1} \zeta(j, k-j) = \zeta(k) \tag{1.2}$$

for $k \in \mathbb{N}$ with $k \geq 3$, where $\zeta(s)$ is the Riemann zeta-function. Equation (1.2) is called the sum formula for double zeta values (see [9]). Research on MZVs has been conducted intensively in this decade (see the survey, [4,13,15]). A recent feature of studies on MZVs is to investigate the structure of the $\mathbb{Q}$-algebra generated by MZVs.

On the other hand, in the late 1990's, it was established that the multiple zeta-function $\zeta(s_1, s_2, ..., s_r)$ of complex variables can be continued meromorphically to the whole complex space $\mathbb{C}^r$ by, for example, Essouabri ([7,8]), Akiyama-Egami-Tanigawa ([1]), Arakawa-Kaneko ([3]), Zhao ([52]) and the second-named author ([23,25]).

Based on these researches, the second-named author raised the following problem several years ago (see, for example, [27]).

Problem. *Are the known relation formulas for multiple zeta values valid only at positive integers, or valid continuously also at other values?*

In other words, is it possible to find certain functional relations for multiple zeta-functions, which include some value-relations for MZVs? A classical example is the following formula which is often called the harmonic product relation:

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

As a related result, Bradley showed a certain class of functional relations called partition identities (see [5]). However, there are many kinds of relations among MZVs, so it is natural to expect that there will be many other classes of functional relations. For example, it seems interesting to prove certain functional relations which include sum formulas for MZVs. In order to give an "answer" to some specific cases of this Problem (the specification being clear from the context), we consider a wider class of multiple zeta-functions as follows.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with rank r. The Witten zeta-function associated with $\mathfrak{g}$ is defined by

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}, \tag{1.3}$$

where the summation runs over all finite dimensional irreducible representations φ of $\mathfrak{g}$.

Witten's motivation [50] for introducing the above zeta-function is to express the volumes of certain moduli spaces in terms of special values of (1.3). The expression is called Witten's volume formula, which especially implies that

$$\zeta_W(2k;\mathfrak{g}) = C_W(2k,\mathfrak{g})\pi^{2kn} \tag{1.4}$$

for any $k \in \mathbb{N}$, where n is the number of all positive roots of $\mathfrak{g}$ and $C_W(2k,\mathfrak{g}) \in \mathbb{Q}$ (Witten [50], Zagier [51]). In their work, the value of $C_W(2k,\mathfrak{g})$ is not explicitly given.

Let Δ be the set of all roots of $\mathfrak{g}$ in the vector space V equipped with an inner product $\langle\cdot,\cdot\rangle$, Δ_+ the set of all positive roots of $\mathfrak{g}$, $\Psi = \{\alpha_1,\ldots,\alpha_r\}$ the fundamental system of Δ, and $\alpha_j^\vee$ the coroot associated with α_j ($1 \leq j \leq r$). Let $\lambda_1,\ldots,\lambda_r$ be the fundamental weights satisfying $\langle\alpha_i^\vee,\lambda_j\rangle = \lambda_j(\alpha_i^\vee) = \delta_{ij}$ (Kronecker's delta). A more explicit form of $\zeta_W(s;\mathfrak{g})$ can be written down in terms of roots and weights by using Weyl's dimension formula (see (1.4) of [18]). Inspired by that form, we introduced in [18] the multi-variable version of Witten zeta-function

$$\zeta_r(\mathbf{s};\mathfrak{g}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha\in\Delta_+} \langle\alpha^\vee, m_1\lambda_1+\cdots+m_r\lambda_r\rangle^{-s_\alpha}, \tag{1.5}$$

where $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^n$. In the case that $\mathfrak{g}$ is of type X_r, we call (1.5) the zeta-function of the root system of type X_r, and denote it by $\zeta_r(\mathbf{s};X_r)$, where $X = A,B,C,D,E,F,G$. We also use the notation $\zeta_W(s;X_r)$ and $C_W(2k,X_r)$, instead of $\zeta_W(s;\mathfrak{g})$ and $C_W(2k,\mathfrak{g})$, respectively. Note that from (1.5) and [18, (1.7)], we have

$$\zeta_W(s;X_r) = K(X_r)^s \zeta_r(s,\ldots,s;X_r), \tag{1.6}$$

where

$$K(X_r) = \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle. \tag{1.7}$$

For example, $K(A_2) = 2$ and $K(C_2) = 6$ (see [18, (2.4) and (2.10)]).

More generally, in [18], we introduced multiple zeta-functions associated with sets of roots. In fact, we studied recursive structures in the family of those zeta-functions, which can be described in terms of Dynkin diagrams of underlying root systems. The meromorphic continuation of those zeta-functions is ensured as a special case of Essouabri's general theorem ([7,8]). It can also be proved by using the Mellin-Barnes integral formula (see [26]).

In [19], we established a general method for evaluating $\zeta_r(s,\ldots,s;X_r)$ at positive integers by considering generalizations of Bernoulli polynomials. In terms of those generalized Bernoulli polynomials, we gave a certain generalization of Witten's volume formula (1.4) with explicit determination of the constant $C_W(2k,X_r)$.

Several cases of zeta-functions of root systems had already been studied. A typical case is of A_2 type:

$$\zeta_2(s_1,s_2,s_3;A_2) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{1.8}$$

In the 1950's, Tornheim [39] first studied the value $\zeta_2(d_1,d_2,d_3;A_2)$ for $d_1,d_2,d_3 \in \mathbb{N}$, which is called the Tornheim double sum. Independently, Mordell [33] studied the value $\zeta_2(2d,2d,2d;A_2)$ ($d \in \mathbb{N}$) and proved, for example,

$$\zeta_2(2,2,2;A_2) = \frac{1}{2835}\pi^6. \tag{1.9}$$

This determines the value of $C_W(2k,A_2)$ in (1.4). Following their works, several value-relations for $\zeta_2(s_1,s_2,s_3;A_2)$ were obtained by several authors (see [6,14,37,40,51]), and also those for its alternating analogues ([41,43,48]). On the other hand, from the analytic viewpoint, the second-named author [24] studied the multi-variable function $\zeta_2(s_1,s_2,s_3;A_2)$ for $s_1,s_2,s_3 \in \mathbb{C}$ which is also called the Mordell-Tornheim double zeta-function, denoted by $\zeta_{MT,2}(s_1,s_2,s_3)$.

Using $\zeta_2(s_1,s_2,s_3;A_2)$, we can give an "answer" to the Problem, that is, functional relations, for example,

$$\zeta(s+1,1) - \zeta_2(s,1,1;A_2) + \zeta(s+2) = 0 \tag{1.10}$$

which holds for all $s \in \mathbb{C}$ except for singularities of the three functions on the left-hand side. In fact, letting $s = k-2$ for $k \geq 3$ in (1.10) and considering partial fraction decompositions, we can obtain the sum formula (1.2). This implies that (1.10) is an answer to the Problem. More generally the third-named author ([47]) proved functional relations for $\zeta_2(s_1, s_2, s_3; A_2)$ which include (1.10) (see Theorem 3.1), and for its alternating analogues ([45]), and its χ-analogues ([46]). A little later, Nakamura gave simple proofs of these results ([34,35]) whose method was inspired by Zagier's lecture.

As for the case of C_2 type, the second-named author defined $\zeta_2(s_1, s_2, s_3, s_4; C_2)$ and studied its analytic properties (see [26]). A little later, the third-named author gave some evaluation formulas for $\zeta_2(k_1, k_2, k_3, k_4; C_2)$ $(k_1, k_2, k_3, k_4 \in \mathbb{N})$ when $k_1 + k_2 + k_3 + k_4$ is odd ([44]).

As for the case of A_3 type, Gunnells and Sczech [10] gave explicit forms of Witten's volume formulas of this type. Recently the second and the third-named authors [30] studied $\zeta_3(\mathbf{s}; A_3)$, and gave certain functional relations for them.

Based on Zagier's work [51] and, in particular, on Nakamura's observation mentioned above, we found that the structural background of those functional relations is given by the symmetry with respect to Weyl groups. From this viewpoint, we considered this structure in [16–19]. In particular, in [19], we gave general forms of functional relations for zeta-functions of root systems. We will recall this result in Section 5.

In the first half of this paper, we summarize known results on functional relations for zeta-functions of root systems, which can be regarded as answers to the Problem. In Section 2, we recall a method of studying relations among Dirichlet series, which is called the 'u-method', introduced in [42]. In Section 3, we summarize known results on functional relations for $\zeta_2(\mathbf{s}; A_2)$. In Section 4, we introduce another method to construct functional relations for multiple Dirichlet series ([31]) which was inspired by Hardy's method of proving the functional equation for $\zeta(s)$ ([11]). In Section 5, we recall general forms of functional relations for $\zeta_r(\mathbf{s}; X_r)$ which we gave in [19]. This is the most general result stated in the present paper, but in general, from this theorem, it is not easy to deduce explicit forms of functional relations in each case. Therefore in the latter half of the paper we give a different method of constructing explicit functional relations. In Section 6, we prove a key lemma (Lemma 6.2) to give a certain procedure to construct functional relations systematically, which has the same flavour as the u-method. In Section 7, by using this lemma combined with a new idea of making use of polylogarithms, we give a functional relation for $\zeta_3(\mathbf{s}; A_3)$ which includes the explicit form of Witten's volume formula of A_3 type. (By "explicit form" we mean that the exact value of $C_W(2k, \mathfrak{g})$ is also determined.) In Sections 8 and 9, by a combination of the methods in Section 4 and in Section 7, we give functional relations for $\zeta_2(\mathbf{s};C_2)$, for $\zeta_3(\mathbf{s};B_3)$, and for $\zeta_3(\mathbf{s};C_3)$ which include explicit forms of Witten's volume formulas.

2. A method to evaluate the Riemann zeta-function

In this section, we introduce a method for evaluating the (multiple) Dirichlet series at positive integers from the information of its trivial zeros, which is called the 'u-method'. In [42], the third-named author first established a method to prove Euler's formula for $\zeta(2k)$ ($k \in \mathbb{N}$). By applying this method to multiple series, several value-relations and functional relations for them have been given (see [40,44–47]). Here we briefly explain this method and recover Euler's formula for $\zeta(s)$:

$$\zeta(2m) = \frac{(-1)^{m-1}2^{2m-1}\pi^{2m}}{(2m)!}B_{2m} \quad (m \in \mathbb{N}), \tag{2.11}$$

where B_n is the nth Bernoulli number defined by $t/(e^t-1) = \sum_{n\geq 0} B_n t^n/n!$. For a small $\delta > 0$ and $u \in [1, 1+\delta]$, we let

$$F(t;u) := \frac{2e^t}{e^t+u} = \sum_{n=0}^{\infty} \mathcal{E}_n(u)\frac{t^n}{n!} \qquad (|t|<\pi), \tag{2.12}$$

where each $\mathcal{E}_n(u)$ is a rational function in u and is continuous for $u \in [1, 1+\delta]$ because $(\partial^k/\partial t^k)F(t;u)$ is continuous for $(t,u) \in \{|t|<\pi\} \times [1, 1+\delta]$. Let $\gamma \in \mathbb{R}$ with $0<\gamma<\pi$, and $\mathcal{C}_\gamma : z = \gamma e^{it}$ for $0 \leq t \leq 2\pi$, where $i = \sqrt{-1}$. From (2.12), we have

$$\int_{\mathcal{C}_\gamma} F(z;u)z^{-n-1}dz = \frac{(2\pi i)\mathcal{E}_n(u)}{n!} \quad (n \in \mathbb{N}_0). \tag{2.13}$$

Let $M = M(\gamma) := \max |F(z,u)|$ for $(z,u) \in \mathcal{C}_\gamma \times [1, 1+\delta]$, which is independent of $u \in [1, 1+\delta]$. Then we obtain

$$\frac{|\mathcal{E}_n(u)|}{n!} \leq \frac{1}{2\pi}\int_{\mathcal{C}_\gamma} |F(z;u)|\ |z|^{-n-1}|dz| \leq \frac{M(\gamma)}{\gamma^n} \quad (n \in \mathbb{N}_0).$$

We let $\phi(s;u) = \sum_{n\geq 1}(-u)^{-n}n^{-s}$ for $s \in \mathbb{C}$. As is well known, $\phi(s;u)$ is convergent for $\Re s > 0$ when $u=1$ and is convergent for any $s \in \mathbb{C}$ when $u>1$. Furthermore, we see that $\phi(s;1) = (2^{1-s}-1)\zeta(s)$. When $u>1$, the second member of (2.12) can be expanded as $-2\sum_{n\geq 1}(-u)^{-n}e^{nt}$. Hence we have $\mathcal{E}_m(u) = -2\phi(-m;u)$ for $m \in \mathbb{N}_0$.

For any $k \in \mathbb{N}$ and $\theta \in (-\pi, \pi)$, we set

$$I_k(\theta; u) := i \sum_{n=1}^{\infty} \frac{(-u)^{-n} \sin(n\theta)}{n^{2k+1}}. \tag{2.14}$$

Suppose $u \in (1, 1+\delta]$ and $\theta \in (-\pi, \pi)$, then

$$\begin{aligned} I_k(\theta; u) &= \sum_{j=0}^{\infty} \phi(2k-2j; u) \frac{(i\theta)^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{k-1} \phi(2k-2j; u) \frac{(i\theta)^{2j+1}}{(2j+1)!} - \frac{1}{2} \sum_{m=0}^{\infty} \mathcal{E}_{2m}(u) \frac{(i\theta)^{2m+2k+1}}{(2m+2k+1)!}. \end{aligned} \tag{2.15}$$

If $|\theta| < \gamma < \pi$, we see that the right-hand side of (2.15) is uniformly convergent with respect to u on $[1, 1+\delta]$, so is continuous on $u \in [1, 1+\delta]$ (see Remark 2.1 (ii)). On the other hand, the left-hand side of (2.15) is also continuous on $u \in [1, 1+\delta]$ from the definition of $I_k(\theta; u)$. Hence we can let $u \to 1$ on both sides of (2.15).

Now we arrive at the crucial point of the argument. We use the fact that $\zeta(-2m) = 0$, that is, $\mathcal{E}_{2m}(1) = 0$ for $m \in \mathbb{N}$ and $\mathcal{E}_0(1) = 1$ (see Remark 2.1 (i)). Then, for $\theta \in (-\pi, \pi)$, we have

$$I_k(\theta; 1) = \sum_{j=0}^{k-1} \phi(2k-2j; 1) \frac{(i\theta)^{2j+1}}{(2j+1)!} - \frac{(i\theta)^{2k+1}}{2(2k+1)!}. \tag{2.16}$$

Since $k \geq 1$, each side of (2.16) is continuous in $\theta \in [-\pi, \pi]$. Hence we can let $\theta \to \pi$ on both sides of (2.16) to obtain

$$0 = I_k(\pi; 1) = \sum_{j=0}^{k-1} \phi(2k-2j; 1) \frac{(i\pi)^{2j+1}}{(2j+1)!} - \frac{(i\pi)^{2k+1}}{2(2k+1)!}.$$

For simplicity, we define

$$\mathcal{A}_{2m} = \phi(2m; 1) \frac{(2m)!}{(i\pi)^{2m}} = (2^{1-2m} - 1)\zeta(2m) \frac{(2m)!}{(i\pi)^{2m}} \quad (m \in \mathbb{N}_0), \tag{2.17}$$

and $\mathcal{A}_0 = -1/2$. Then (2.16) implies that

$$\sum_{j=0}^{k} \binom{2k+1}{2j+1} \mathcal{A}_{2k-2j} = 0$$

for $k \in \mathbb{N}$. Since $\mathcal{A}_0 = -1/2$, we obtain

$$-\frac{t}{2} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{2k+1}{2j+1} \mathcal{A}_{2k-2j} \right) \frac{t^{2k+1}}{(2k+1)!} = \left(\sum_{m=0}^{\infty} \mathcal{A}_{2m} \frac{t^{2m}}{(2m)!} \right) \frac{e^t - e^{-t}}{2}.$$

We can easily check that

$$\frac{2t}{e^t - e^{-t}} = \frac{2te^t}{e^{2t}-1} = \sum_{m=0}^{\infty} \left(2 - 2^{2m}\right) B_{2m} \frac{t^{2m}}{(2m)!},$$

so we have $\mathcal{A}_{2m} = (2^{2m-1} - 1)B_{2m}$ for any nonnegative integer m. In view of (2.17), we obtain Euler's formula (2.11).

Remark 2.1. (i) It should be noted that the fact

$$-2\phi(-2m; 1) \left(= -2\left(2^{2m+1} - 1\right)\zeta(-2m)\right) = \mathcal{E}_{2m}(1)\,(= 0) \quad (m \in \mathbb{N}) \tag{2.18}$$

(the trivial zeros of the zeta-function!) plays a vital role in the above argument. In fact, equation (2.18) can be obtained by proving

$$\begin{aligned}\sum_{n=0}^{\infty} \{-2\phi(-n; 1)\} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left\{-2\left(2^{n+1} - 1\right)\zeta(-n)\right\} \frac{t^n}{n!} \\ = 1 + 2\sum_{n=1}^{\infty} \left(2^{n+1} - 1\right) B_{n+1} \frac{t^n}{(n+1)!} &= 2 + \frac{4}{e^{2t}-1} - \frac{2}{e^t - 1} = \frac{2e^t}{e^t + 1},\end{aligned} \tag{2.19}$$

because $\zeta(1-k) = -B_k/k$ ($k \in \mathbb{N}$; $k \geq 2$) and $\zeta(0) = -1/2$. (ii) Also we note that $\phi(s; u)$ is continuous in u as $u \to 1 + 0$ for any $s \in \mathbb{C}$. In fact, similarly to the case of $\zeta(s)$, we can easily see that

$$\phi(s; u) = \frac{1}{(e^{2\pi i s} - 1)\,\Gamma(s)} \int_C \frac{e^t}{e^t + u} t^{s-1} dt, \tag{2.20}$$

where C is the contour, that is, the path which starts at $+\infty$, passes through the real axis, goes around the origin counterclockwise and goes back to $+\infty$. From (2.20), we immediately obtain the desired continuity.

3. Functional relations for $\zeta_2(s_1, s_2, s_3; A_2)$

By applying the method introduced in Section 2 to $\zeta_2(s_1, s_2, s_3; A_2)$, the third-named author gave value-relation formulas for $\zeta_2(s_1, s_2, s_3; A_2)$ (see [40]). Moreover, applying the above method to the double series in complex variables, he gave functional relations for $\zeta_2(s_1, s_2, s_3; A_2)$ (see [47]). The original form in [47, Theorem 4.5] is a little complicated. By using a certain transformation formula (see Lemma 6.1, which is [28, Lemma 2.1]), we obtain the following simpler form.

Theorem 3.1. *For $k, l \in \mathbb{N}_0$,*

$$\begin{aligned} &\zeta_2(k,l,s;A_2) + (-1)^k \zeta_2(k,s,l;A_2) + (-1)^l \zeta_2(l,s,k;A_2) \\ &= 2\sum_{\rho=0}^{[k/2]} \binom{k+l-2\rho-1}{l-1} \zeta(2\rho)\zeta(s+k+l-2\rho) \\ &+ 2\sum_{\rho=0}^{[l/2]} \binom{k+l-2\rho-1}{k-1} \zeta(2\rho)\zeta(s+k+l-2\rho) \end{aligned} \tag{3.21}$$

holds for all $s \in \mathbb{C}$ except for singularities of functions on both sides.

Proof. For $\theta, r, u \in \mathbb{R}$ with $r > 1$ and $u \in [1, 1+\delta]$, and $k, p \in \mathbb{N}_0$, we let

$$\mathfrak{F}(i\theta;r;u) = \sum_{n=1}^{\infty} \frac{(-u)^{-n} e^{int}}{n^r},$$

$$\begin{aligned} \mathcal{J}_p(i\theta;k;u) = &\frac{i^{p-1}}{2} \left\{ \mathfrak{F}(i\theta;k;u) + (-1)^{p-1} \mathfrak{F}(-i\theta;k;u) \right\} \\ &- \sum_{j=0}^{k} \phi(k-j;u)\ \varepsilon_{p+1+j} \frac{(i\theta)^j}{j!}, \end{aligned}$$

where $\varepsilon_m = \{1 + (-1)^m\}/2$ for $m \in \mathbb{Z}$. Then, similarly to the proof of (2.16), we see that if $k \not\equiv p \pmod 2$ and $\theta \in (-\pi, \pi)$ then $\mathcal{J}_p(i\theta;k;u) \to 0$ as $u \to 1$. Let

$$\begin{aligned} R(s_1,s_2;s_3;u) &= \sum_{m,n=1}^{\infty} \frac{(-u)^{-2m-n}}{m^{s_1} n^{s_2} (m+n)^{s_3}}, \\ S(s_1,s_2;s_3;u) &= \sum_{m,n=1}^{\infty} \frac{(-u)^{-m-n}}{m^{s_1} n^{s_2} (m+n)^{s_3}}, \end{aligned}$$

which are double analogues of $\phi(s;u)$. Then, for $u\in(1,1+\delta]$,

$$\begin{aligned}\mathcal{J}_p(i\theta;k;u)\mathfrak{F}(i\theta;r;u) &= i^{p-1}\sum_{N=0}^{\infty}\frac{1}{2}\Big\{S(k,r;-N;u)+(-1)^{p-1}R(k,-N;r;u)\\ &\qquad +(-1)^{p-1+N}R(r,-N;k;u)\Big\}\frac{(i\theta)^N}{N!}\\ -\sum_{N=0}^{\infty}\sum_{j=0}^{k}\binom{N}{j}&\phi(k-j;u)\phi(r+j-N;u)\varepsilon_{p+1+j}\frac{(i\theta)^N}{N!}+\frac{(-i)^{p-1}}{2}\sum_{m=1}^{\infty}\frac{u^{-2m}}{m^{k+r}}.\end{aligned}$$

As noted above, the left-hand side tends to 0 as $u\to 1$ when $k\not\equiv p \pmod 2$ and $\theta\in(-\pi,\pi)$. Therefore, similarly to the case of $\zeta(s)$, we can obtain the original form of the functional relation ([47, Lemma 4.5]):

$$\begin{aligned}&\zeta_2(k,l,s;A_2)+(-1)^k\zeta_2(k,s,l;A_2)+(-1)^l\zeta_2(l,s,k;A_2)\\ &=2\sum_{\substack{j=0\\ j\equiv k\ (2)}}^{k}\left(2^{1-k+j}-1\right)\zeta(k-j)\\ &\quad\times\sum_{\mu=0}^{[j/2]}\frac{(i\pi)^{2\mu}}{(2\mu)!}\binom{l-1+j-2\mu}{j-2\mu}\zeta(l+j+s-2\mu)\\ &\quad-4\sum_{\substack{j=0\\ j\equiv k\ (2)}}^{k}\left(2^{1-k+j}-1\right)\zeta(k-j)\sum_{\mu=0}^{[(j-1)/2]}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\sum_{\substack{\nu=0\\ \nu\equiv l\ (2)}}^{l}\zeta(l-\nu)\\ &\quad\times\binom{\nu-1+j-2\mu}{j-2\mu-1}\zeta(\nu+j+s-2\mu)\end{aligned}$$

holds for all $s\in\mathbb{C}$ except for singularities of functions on both sides, where $k,l\in\mathbb{N}$. Additionally, using a transformation formula in Lemma 6.1 below ([28, Lemma 2.1]), we obtain (3.21). □

Example 3.1. Setting $(k,l)=(2,2),\ (3,2)$ in (3.21), we have

$$\zeta_2(2,2,s;A_2)+2\zeta_2(2,s,2;A_2)=4\zeta(2)\zeta(s+2)-6\zeta(s+4), \tag{3.22}$$

$$\zeta_2(3,s,2;A_2)-\zeta_2(3,2,s;A_2)-\zeta_2(2,s,3;A_2)=10\zeta(s+5)-6\zeta(2)\zeta(s+3). \tag{3.23}$$

Setting $s=2$ in (3.22) and (3.23), we have (1.9) and

$$\zeta_2(2,2,3;A_2)=6\zeta(2)\zeta(5)-10\zeta(7), \tag{3.24}$$

respectively, where (3.24) was given by Tornheim [39]. Note that $\zeta_2(k,0,l;A_2)=\zeta(l,k)$. Then, setting $s=0$ in (3.23), we have $\zeta(2,3)-\zeta(3,2)=10\zeta(5)-5\zeta(2)\zeta(3)$. On the other hand, it is well-known that $\zeta(3,2)+\zeta(2,3)=\zeta(2)\zeta(3)-\zeta(5)$. Combining these results, we obtain the known results

$$\zeta(2,3)=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3); \quad \zeta(3,2)=-\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3),$$

which were originally obtained by double shuffle relations.

Remark 3.1. In [32], the second and the third-named authors generalized the result in Theorem 3.1 to the case of polylogarithmic analogues, that is,

$$\begin{aligned}\sum_{m,n=1}^{\infty}\frac{x^n}{m^k n^l (m+n)^s}+(-1)^k\sum_{m,n=1}^{\infty}\frac{x^n}{m^k n^s (m+n)^l}+(-1)^l\sum_{m,n=1}^{\infty}\frac{x^{m+n}}{m^l n^s (m+n)^k}\\ =2\sum_{\rho=0}^{[k/2]}\binom{k+l-2\rho-1}{k-2\rho}\zeta(2\rho)\sum_{m=1}^{\infty}\frac{x^m}{m^{s+k+l-2\rho}}\\ +2\sum_{\rho=0}^{[l/2]}\binom{k+l-2\rho-1}{l-2\rho}\zeta(2\rho)\sum_{m=1}^{\infty}\frac{x^m}{m^{s+k+l-2\rho}},\end{aligned} \tag{3.25}$$

for $x\in\mathbb{C}$ with $|x|\le 1$. The idea of this generalization gives an important key to construct functional relations for zeta-functions of the type of A_3, C_2, B_3 and C_3 (see Remark 7.1).

In [34], Nakamura gave an alternative simple proof of (3.21) whose method was inspired by Zagier's lecture. We explain this method. We denote by $\{B_n(x)\}$ the Bernoulli polynomials defined by $te^{xt}/(e^t-1)=\sum_{n\ge 0}B_n(x)t^n/n!$ ($|t|<2\pi$). It is known (see [2, p.266 - p.267]) that $B_{2j}(0)=B_{2j}$ for $j\in\mathbb{N}_0$ and

$$B_j(x-[x])=-\frac{j!}{(2\pi i)^j}\lim_{K\to\infty}\sum_{\substack{k=-K\\ k\ne 0}}^{K}\frac{e^{2\pi ikx}}{k^j} \qquad (j\in\mathbb{N}), \tag{3.26}$$

where $[\cdot]$ is the integer part. For $k\in\mathbb{Z}$, $j\in\mathbb{N}$ we have

$$\int_0^1 e^{-2\pi ikx}B_j(x)\,dx=\begin{cases}0 & (k=0),\\ -(2\pi ik)^{-j}j! & (k\ne 0),\end{cases} \tag{3.27}$$

by (3.26). We further quote [2, p.276 19.(b)], for $p, q \geq 1$, which is

$$B_p(x)B_q(x) = \sum_{k=0}^{\max(p,q)/2} \left\{ p\binom{q}{2k} + q\binom{p}{2k} \right\} \frac{B_{2k}B_{p+q-2k}(x)}{p+q-2k} - (-1)^p \frac{p!q!}{(p+q)!} B_{p+q}. \tag{3.28}$$

On the other hand, for $a, b \geq 2$, and $\Re(s) > 1$, we have

$$\int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l x}}{l^a} \sum_{m=1}^{\infty} \frac{e^{2\pi i m x}}{m^b} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{n^s} dx = \zeta_2(a, b, s; A_2),$$

$$\int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{2\pi i m x}}{(m+l)^a l^b} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n x}}{n^s} dx = \zeta_2(b, s, a; A_2),$$

$$\int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi i l x}}{l^{a+b-j}} \sum_{m=1}^{\infty} \frac{e^{-2\pi i n x}}{n^s} dx = \zeta(a+b+s-j).$$

Combining these relations and (3.26)-(3.28), we see that

$$\begin{aligned} &\zeta_2(a, b, s; A_2) + (-1)^b \zeta_2(b, s, a; A_2) + (-1)^a \zeta_2(s, a, b; A_2) \\ &= \frac{2}{a!b!} \sum_{k=0}^{\max(a,b)/2} \left\{ a\binom{b}{2k} + b\binom{a}{2k} \right\} \\ &\quad \times (a+b-2k-1)!(2k)!\zeta(2k)\zeta(a+b-s-2k), \end{aligned} \tag{3.29}$$

which coincides with (3.21). Nakamura also gave some more generalized formulas for double zeta and L-functions and triple zeta-functions of Mordell and Tornheim type ([35,36]). Furthermore triple zeta and L-functions were studied by Nakamura, Ochiai, and the second and the third-named authors ([28,29]). The aforementioned Lemma 6.1 first appeared in those studies.

4. Another method to construct functional relations for Dirichlet series

In this section, we introduce another method to study functional relations for Dirichlet series, whose basic idea was originally introduced by Hardy. Hardy gave an alternative proof of the functional equation for $\zeta(s)$ ([11], see also [38] Section 2.2). By generalizing this method, we can give functional relations for multiple zeta-functions, for example, (3.21) in Theorem 3.1.

First we consider a general Dirichlet series $Z(s)=\sum_{m=1}^{\infty} a_m m^{-s}$ where $\{a_n\}\subset\mathbb{C}$. Let $\Re s=\rho$ $(\rho\in\mathbb{R})$ be the abscissa of convergence of $Z(s)$. This means that if $\Re s>\rho$ then $Z(s)$ is convergent and if $\Re s<\rho$ then $Z(s)$ is not convergent. We further assume that $0\le\rho<1$.

Theorem 4.1 ([31], Theorem 3.1). *Assume that* $\sum_{m=1}^{\infty} a_m\sin(mt)=0$ *is boundedly convergent for* $t>0$ *and that, for* $\rho<s<1$,

$$\lim_{\lambda\to\infty}\sum_{m=1}^{\infty} a_m\int_{\lambda}^{\infty} t^{s-1}\sin(mt)dt=0. \tag{4.30}$$

Then $Z(s)$ *can be continued meromorphically to* $\mathbb{C}$*, and actually* $Z(s)=0$ *for all* $s\in\mathbb{C}$*. The same conclusion holds if we assume the formulas similar to the above but "*sin*" (two places) is replaced by "*cos*".*

We give a simple example showing how to apply this theorem. From (2.16) and the formula obtained by differentiating both sides of (2.16), we have

$$\sum_{l=1}^{\infty}\frac{(-1)^l\cos(l\theta)}{l^{2p}}=\sum_{\nu=0}^{p}\ \phi(2p-2\nu)\ \frac{(-1)^{\nu}\theta^{2\nu}}{(2\nu)!}, \tag{4.31}$$

$$\sum_{l=1}^{\infty}\frac{(-1)^l\sin(l\theta)}{l^{2q+1}}=\sum_{\nu=0}^{q}\ \phi(2q-2\nu)\ \frac{(-1)^{\nu}\theta^{2\nu+1}}{(2\nu+1)!} \tag{4.32}$$

for $p\in\mathbb{N}$, $q\in\mathbb{N}_0$ and $\theta\in(-\pi,\pi)$. Note that the case $q=0$ in (4.32) is a little delicate. To prove this case, we define $I_0(\theta;u)$ for $\theta\in(-\pi,\pi)$ and $u\in[1,1+\delta]$ by (2.14). Then equation (2.15) in the case $q=0$ holds for $u\in(1,1+\delta]$. From [49, § 3.35] (see also [31, Lemma 4.1]) and Abel's theorem (see [49, § 3.71]), we can let $u\to1$ in (2.15) for $I_0(\theta;u)$. Then, as well as (2.16), we obtain the case $q=0$ in (4.32). Additionally we note that if $p,q\in\mathbb{N}$ then (4.31) and (4.32) hold for $\theta\in[-\pi,\pi]$ because both sides are continous for $\theta\in[-\pi,\pi]$.

Combining these results and putting $t=\theta+\pi$, we obtain, for $t\in\mathbb{R}\backslash2\pi\mathbb{Z}$,

$$\begin{aligned}\sum_{m,n=1}^{\infty}\frac{\cos((m+n)t)}{m^2n^2}+2\sum_{m,n=1}^{\infty}\frac{\cos(mt)}{n^2(m+n)^2}\\ +6\sum_{m=1}^{\infty}\frac{\cos(mt)}{m^4}-4\zeta(2)\sum_{m=1}^{\infty}\frac{\cos(mt)}{m^2}=0\end{aligned} \tag{4.33}$$

(see [31, Lemma 2.2]). We denote by $f(t)$ the left-hand side of (4.33). Note that each sum on the left-hand side of (4.33) is absolutely and uniformly convergent for $t \in \mathbb{R}$. Hence, for $s \in \mathbb{R}$ with $0 < s < 1$, we have

$$\begin{aligned} 0 &= \int_0^\infty t^{s-1} f(t) dx \\ &= \int_0^\infty t^{s-1} \Big\{ \sum_{m,n=1}^\infty \frac{\cos((m+n)t)}{m^2 n^2} + 2 \sum_{m,n=1}^\infty \frac{\cos(mt)}{n^2 (m+n)^2} \\ &\quad + 6 \sum_{m=1}^\infty \frac{\cos(mt)}{m^4} - 4\zeta(2) \sum_{m=1}^\infty \frac{\cos(mt)}{m^2} \Big\} dt. \end{aligned} \tag{4.34}$$

By the same argument as in [38, Section 2.1], we have

$$\int_\lambda^\infty \frac{\cos(Nx)}{x^{1-s}} dx = \left[\frac{\sin(Nx)}{Nx^{1-s}} \right]_\lambda^\infty - \frac{s-1}{N} \int_\lambda^\infty \frac{\sin(Nx)}{x^{2-s}} dx = O\left(\frac{1}{N\lambda^{1-s}} \right)$$

for $N \in \mathbb{N}$. Using this result, we see that

$$\begin{aligned} &\lim_{\lambda \to \infty} \sum_{m,n=1}^\infty \frac{1}{m^2 n^2} \int_\lambda^\infty \frac{\cos((m+n)x)}{x^{1-s}} dx = 0, \\ &\lim_{\lambda \to \infty} \sum_{m,n=1}^\infty \frac{1}{n^2 (m+n)^2} \int_\lambda^\infty \frac{\cos(mx)}{x^{1-s}} dx = 0, \\ &\lim_{\lambda \to \infty} \sum_{m=1}^\infty \frac{1}{m^l} \int_\lambda^\infty \frac{\cos(mx)}{x^{1-s}} dx = 0 \quad (l = 2, 4) \end{aligned}$$

hold for $0 < s < 1$. Hence we can justify term-by-term integration on the right-hand side of (4.34). Therefore it follows from Theorem 4.1 and the facts

$$\begin{aligned} \int_0^\infty \frac{\cos bx}{x^{1-s}} dx &= \frac{\pi}{2} b^{-s} \frac{\sec(\pi(1-s)/2)}{\Gamma(1-s)}, \\ \int_0^\infty \frac{\sin bx}{x^{1-s}} dx &= \frac{\pi}{2} b^{-s} \frac{\operatorname{cosec}(\pi(1-s)/2)}{\Gamma(1-s)} \end{aligned}$$

for $b > 0$ and $0 < s < 1$ (see [49, Chapter 12]) that the functional relation

$$\zeta_2(2, 2, s; A_2) + 2\zeta_2(s, 2, 2; A_2) + 6\zeta(s+4) - 4\zeta(2)\zeta(s+2) = 0 \tag{4.35}$$

holds for $0 < s < 1$ (and then for any $s \in \mathbb{C}$ by analytic continuation). This coincides with (3.22).

By using this method, we can give functional relations for more general types of multiple zeta-functions (see [31]).

5. A general form of functional relations

In the previous sections, we present various methods to obtain functional relations. However in those methods, it is not clear why these functional relations exist. From the viewpoint of Weyl group symmetry in the underlying Lie algebra structure, we can give a certain explanation of this phenomenon. In fact, in view of the Weyl group symmetry, we can show a general form of functional relations for zeta-functions of root systems. For the details, see [19,22].

First we prepare some notation. Let V be an r-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Let Δ be a finite reduced root system in V of X_r type and $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ its fundamental system. Let Δ_+ and Δ_- be the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta = \Delta_+ \coprod \Delta_-$. Let $Q^\vee$ be the coroot lattice, P the weight lattice, P_+ the set of integral dominant weights and P_{++} the set of integral strongly dominant weights respectively defined by

$$Q^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\,\alpha_i^\vee, \qquad P = \bigoplus_{i=1}^{r} \mathbb{Z}\,\lambda_i, \qquad P_+ = \bigoplus_{i=1}^{r} \mathbb{N}_0\,\lambda_i, \qquad P_{++} = \bigoplus_{i=1}^{r} \mathbb{N}\,\lambda_i, \tag{5.36}$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ form a basis dual to $\Psi^\vee$ satisfying $\langle \alpha_i^\vee, \lambda_j \rangle = \delta_{ij}$. The reflection $\sigma_\alpha : V \to V$ with respect to a root $\alpha \in \Delta$ is defined by $\sigma_\alpha(v) = v - \langle \alpha^\vee, v \rangle \alpha$. For a subset $A \subset \Delta$, let $W(A)$ be the group generated by reflections σ_α for $\alpha \in A$. Let $W = W(\Delta)$ be the Weyl group. Then $\sigma_j = \sigma_{\alpha_j}$ $(1 \leq j \leq r)$ generates W. We denote the fundamental domain called the fundamental Weyl chamber by $C = \{v \in V \mid \langle \Psi^\vee, v \rangle \geq 0\}$, where $\langle \Psi^\vee, v \rangle$ means any of $\langle \alpha^\vee, v \rangle$ for $\alpha^\vee \in \Psi^\vee$. Then W acts on the set of Weyl chambers $WC = \{wC \mid w \in W\}$ simply transitively. Moreover if $wx = y$ for $x, y \in C$, then $x = y$ holds. The stabilizer W_x of a point $x \in V$ is generated by the reflections which stabilize x. We see that $P_+ = P \cap C$. For $w \in W$, we set $\Delta_w = \Delta_+ \cap w^{-1}\Delta_-$.

Let $I \subset \{1, \ldots, r\}$ and $\Psi_I = \{\alpha_i \mid i \in I\} \subset \Psi$. Let V_I be the linear subspace spanned by Ψ_I. Then $\Delta_I = \Delta \cap V_I$ is a root system in V_I whose fundamental system is Ψ_I. For the root system Δ_I, we denote the corresponding coroot lattice (resp. weight lattice etc.) by $Q_I^\vee = \bigoplus_{i \in I} \mathbb{Z}\,\alpha_i^\vee$ (resp. $P_I = \bigoplus_{i \in I} \mathbb{Z}\,\lambda_i$ etc.). Let $\Delta_+^\vee$ be the set of all positive coroots, and $W^I = \{w \in W \mid \Delta_{I+}^\vee \subset w\Delta_+^\vee\}$.

Let $\mathbf{y} \in V$ and $\mathbf{s} = (s_\alpha)_{\alpha \in \overline{\Delta}} \in \mathbb{C}^{|\Delta_+|}$, where $\overline{\Delta}$ is the quotient of Δ obtained by identifying α and $-\alpha$. Define an action of W to $\mathbf{s}$ by $(w\mathbf{s})_\alpha =$

$s_{w^{-1}\alpha}$. Now we introduce the "twisted" multiple zeta-function of the form

$$\zeta_r(\mathbf{s},\mathbf{y};\Delta) = \sum_{\lambda\in P_{++}} e^{2\pi\sqrt{-1}\langle \mathbf{y},\lambda\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle \alpha^\vee,\lambda\rangle^{s_\alpha}}. \tag{5.37}$$

A motivation of introducing such a generalized form with exponential factors is to study multiple L-functions of root systems (see [17,21]). When $\mathbf{y}=0$ in (5.37), the function $\zeta_r(\mathbf{s};\Delta)=\zeta_r(\mathbf{s},0;\Delta)$ coincides with the zeta-function of the root system Δ, defined by (1.5).

For $s\in\mathbb{C}$, $\Re s>1$ and $x,c\in\mathbb{R}$, let

$$\mathcal{L}_s(x,c) = -\frac{\Gamma(s+1)}{(2\pi\sqrt{-1})^s} \sum_{\substack{n\in\mathbb{Z}\\ n+c\neq 0}} \frac{e^{2\pi\sqrt{-1}(n+c)x}}{(n+c)^s}. \tag{5.38}$$

Then we obtain the following general form of functional relations for zeta-functions of root systems.

Theorem 5.1 ([19], Theorem 4.3). *When $I\neq\emptyset$, for $\mathbf{s}\in\mathcal{S}$ and $\mathbf{y}\in V$, we have*

$$\begin{aligned} &S(\mathbf{s},\mathbf{y};I;\Delta) \qquad\qquad (5.39)\\ &:= \sum_{w\in W^I}\Big(\prod_{\alpha\in\Delta_{w^{-1}}}(-1)^{-s_\alpha}\Big)\zeta_r(w^{-1}\mathbf{s},w^{-1}\mathbf{y};\Delta)\\ &= (-1)^{|\Delta_+\setminus\Delta_{I+}|}\Big(\prod_{\alpha\in\Delta_+\setminus\Delta_{I+}}\frac{(2\pi\sqrt{-1})^{s_\alpha}}{\Gamma(s_\alpha+1)}\Big)\sum_{\lambda\in P_{I++}} e^{2\pi\sqrt{-1}\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_{I+}}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}\\ &\quad\times\int_0^1\cdots\int_0^1 \exp\Big(-2\pi\sqrt{-1}\sum_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)} x_\alpha\langle\alpha^\vee,\lambda\rangle\Big)\Big(\prod_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)}\mathcal{L}_{s_\alpha}(x_\alpha,0)\Big)\\ &\quad\times\Big(\prod_{i\in I^c}\mathcal{L}_{s_{\alpha_i}}\Big(\langle\mathbf{y},\lambda_i\rangle-\sum_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)} x_\alpha\langle\alpha^\vee,\lambda_i\rangle,0\Big)\Big)\prod_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)} dx_\alpha. \end{aligned}$$

Remark 5.1. We also studied the case $I=\emptyset$ and gave an integral expression of $S(\mathbf{s},\mathbf{y};\emptyset;\Delta)$ similar to (5.39) (see [19, Theorem 4.4]).

Example 5.1. Here we give an alternative proof of (3.23). Set $\Delta_+ = \Delta_+(A_2) = \{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$, and $\mathbf{y}=0$, $\mathbf{s}=(2,s,3)$ for $s\in\mathbb{C}$ with

$\Re s > 1$, $I = \{2\}$, that is, $\Delta_{I+} = \{\alpha_2\}$. Then we see that the left-hand side of (5.39) is

$$S(\mathbf{s}, \mathbf{y}; I; \Delta) = \sum_{m,n=1}^{\infty} \frac{1}{m^2 n^s (m+n)^3} - \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{1}{m^2 n^s (-m+n)^3}$$
$$= \zeta_2(2, s, 3; A_2) - \zeta_2(3, 2, s; A_2) + \zeta_2(3, s, 2; A_2).$$

On the other hand, the right-hand side of (5.39) is

$$\left(\frac{(2\pi\sqrt{-1})^2}{2!}\right)\left(\frac{(2\pi\sqrt{-1})^3}{3!}\right) \sum_{m=1}^{\infty} \frac{1}{m^s} \int_0^1 e^{-2\pi\sqrt{-1}mx} \mathcal{L}_2(x,0)\mathcal{L}_3(-x,0)dx$$
$$= \left(\frac{(2\pi\sqrt{-1})^2}{2!}\right)\left(\frac{(2\pi\sqrt{-1})^3}{3!}\right) \sum_{m=1}^{\infty} \frac{1}{m^s} \int_0^1 e^{-2\pi\sqrt{-1}mx} B_2(x)B_3(1-x)dx,$$

by $\mathcal{L}_k(x,0) = B_k(x-[x])$ for $x \in \mathbb{R}$ (see (3.26)). Hence, by using (3.27) and (3.28), we obtain (3.23).

6. Some lemmas for explicit construction of functional relations

From the general form of functional relations in Theorem 5.1, it is possible to deduce explicit formulas of functional relations for zeta-functions of root systems, e.g., as in Example 5.1. However, if a rank of the root system is high, then it seems quite hard to give explicit forms directly from Theorem 5.1. Therefore now we introduce a different procedure to construct explicit functional relations. For this aim, we give some general preparatory lemmas. We first quote the following lemma from our previous paper. Let $\phi(s) := \sum_{n\geq 1}(-1)^n n^{-s} = \left(2^{1-s}-1\right)\zeta(s)$, and $\varepsilon_\nu := (1+(-1)^\nu)/2$ ($\nu \in \mathbb{Z}$).

Lemma 6.1 ([28] Lemma 2.1). *Let $f, g : \mathbb{N}_0 \to \mathbb{C}$ be arbitrary functions. Then, for $a \in \mathbb{N}$, we have*

$$\sum_{k=0}^{a} \phi(a-k)\varepsilon_{a-k} \sum_{\mu=0}^{[k/2]} f(k-2\mu)\frac{(-1)^\mu \pi^{2\mu}}{(2\mu)!} = \sum_{\xi=0}^{[a/2]} \zeta(2\xi) f(a-2\xi), \quad (6.40)$$

and

$$\sum_{k=1}^{a} \phi(a-k)\varepsilon_{a-k} \sum_{\mu=0}^{[(k-1)/2]} g(k-2\mu)\frac{(-1)^\mu \pi^{2\mu}}{(2\mu+1)!} = -\frac{1}{2}g(a). \quad (6.41)$$

Corollary 6.1. *With the same notation as in Lemma 6.1, put*

$$h(d) := \sum_{\mu=0}^{[d/2]} g(d-2\mu)\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!} \qquad (d \in \mathbb{N}_0).$$

Then we have

$$g(d) = -2\sum_{\mu=0}^{d} \phi(d-\mu)\varepsilon_{d-\mu}h(\mu) \qquad (d \in \mathbb{N}_0).$$

Proof. In (6.41), we replace $g(x)$ by $g(x-1)$. Then (6.41) implies that

$$\sum_{k=1}^{a}\phi(a-k)\varepsilon_{a-k}h(k-1) = -\frac{1}{2}g(a-1)$$

for $a \in \mathbb{N}$. Replacing a by $d+1$ and $k-1$ by μ, respectively, we obtain the desired assertion. □

Using Lemma 6.1, we prove the following lemma which is a key to construct functional relations. Let $h \in \mathbb{N}$, and

$$\mathfrak{C} := \{C(l) \in \mathbb{C} \,|\, l \in \mathbb{Z},\ l \neq 0\},$$
$$\mathfrak{D} := \{D(N;m;\eta) \in \mathbb{R} \,|\, N,m,\eta \in \mathbb{Z},\ N \neq 0,\ m \geq 0,\ 1 \leq \eta \leq h\},$$
$$\mathfrak{A} := \{a_\eta \in \mathbb{N} \,|\, 1 \leq \eta \leq h\}$$

be sets of numbers indexed by integers. We let

$$\binom{x}{k} := \begin{cases} \frac{x(x-1)\cdots(x-k+1)}{k!} & (k \in \mathbb{N}), \\ 1 & (k = 0). \end{cases}$$

Lemma 6.2. *With the above notation, we assume that the infinite series appearing in*

$$\sum_{\substack{N\in\mathbb{Z}\\ N\neq 0}}(-1)^N C(N)e^{iN\theta} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta-k)\varepsilon_{a_\eta-k} \times \sum_{\xi=0}^{k}\left\{\sum_{\substack{N\in\mathbb{Z}\\ N\neq 0}}(-1)^N D(N;k-\xi;\eta)e^{iN\theta}\right\}\frac{(i\theta)^{\xi}}{\xi!} \tag{6.42}$$

are absolutely convergent for $\theta \in [-\pi,\pi]$, and that (6.42) is a constant function for $\theta \in [-\pi,\pi]$. Then, for $d \in \mathbb{N}_0$,

$$\begin{aligned}
&\sum_{\substack{N\in\mathbb{Z}\\ N\neq 0}} \frac{(-1)^N C(N) e^{iN\theta}}{N^d} = 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta} \phi(a_\eta - k)\varepsilon_{a_\eta - k} \qquad (6.43)\\
&\quad\times \sum_{\xi=0}^{k}\left\{\sum_{\omega=0}^{k-\xi}\binom{\omega+d-1}{\omega}(-1)^\omega \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\xi-\omega;\eta)e^{im\theta}}{m^{d+\omega}}\right\}\frac{(i\theta)^\xi}{\xi!}\\
&- 2\sum_{k=0}^{d}\phi(d-k)\varepsilon_{d-k}\sum_{\xi=0}^{k}\left\{\sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta-1}\binom{\omega+k-\xi}{\omega}(-1)^\omega\right.\\
&\qquad\qquad \left.\times \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;a_\eta-1-\omega;\eta)}{m^{k-\xi+\omega+1}}\right\}\frac{(i\theta)^\xi}{\xi!}
\end{aligned}$$

holds for $\theta \in [-\pi,\pi]$, where the infinite series appearing on both sides of (6.43) are absolutely convergent for $\theta \in [-\pi,\pi]$.

Proof. For $d \in \mathbb{N}_0$, put

$$\begin{aligned}
&G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) \qquad (6.44)\\
&:= \frac{1}{i^d}\left[\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l C(l)e^{il\theta}}{l^d} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta-k)\varepsilon_{a_\eta-k}\right.\\
&\quad\left.\times\sum_{\xi=0}^{k}\left\{\sum_{\nu=0}^{\xi}\binom{d-1+\xi-\nu}{\xi-\nu}(-1)^\xi\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\xi;\eta)e^{im\theta}}{m^{d+\xi-\nu}}\frac{(-i\theta)^\nu}{\nu!}\right\}\right]\\
&= \frac{1}{i^d}\left[\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l C(l)e^{il\theta}}{l^d} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta-k)\varepsilon_{a_\eta-k}\right.\\
&\quad\left.\times\sum_{\nu=0}^{k}\left\{\sum_{\omega=0}^{k-\nu}\binom{d-1+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\nu-\omega;\eta)e^{im\theta}}{m^{d+\omega}}\right\}\frac{(i\theta)^\nu}{\nu!}\right].
\end{aligned}$$

Note that the second equality of (6.44) follows by putting $\omega = \xi - \nu$. Then the assumption of (6.42) implies that

$$G_0(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) = R_0 \quad (\theta \in [-\pi,\pi]), \qquad (6.45)$$

where there exists a constant $R_0 = R_0(\mathfrak{C};\mathfrak{D};\mathfrak{A}) \in \mathbb{C}$, because $\binom{-1+\xi-\nu}{\xi-\nu} = 0$ if $\xi > \nu$. Furthermore we can check that $G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A})$ is absolutely convergent with respect to $\theta \in [-\pi,\pi]$ and that

$$\frac{d}{d\theta}G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) = G_{d-1}(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) \quad (d \in \mathbb{N}). \tag{6.46}$$

In fact, if we differentiate the second member of (6.44) with respect to θ, then we have

$$\begin{aligned}
&\frac{d}{d\theta}G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) \\
&= \frac{1}{i^{d-1}}\Bigg[\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l C(l)e^{il\theta}}{l^{d-1}} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta - k)\varepsilon_{a_\eta-k} \\
&\quad\times\sum_{\xi=0}^{k}\Bigg\{\sum_{\nu=0}^{\xi}\binom{d-1+\xi-\nu}{\xi-\nu}(-1)^\xi\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\xi;\eta)e^{im\theta}}{m^{d+\xi-\nu-1}}\frac{(-i\theta)^\nu}{\nu!} \\
&\quad-\sum_{\nu=1}^{\xi}\binom{d-1+\xi-\nu}{\xi-\nu}(-1)^\xi\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\xi;\eta)e^{im\theta}}{m^{d+\xi-\nu}}\frac{(-i\theta)^{\nu-1}}{(\nu-1)!}\Bigg\}\Bigg].
\end{aligned}$$

Replacing $\nu-1$ by μ in the last member of the above summation, and using the well-known relation

$$-\binom{m-1}{l-1}+\binom{m}{l} = \binom{m-1}{l} \quad (l,m \in \mathbb{N}),$$

we obtain (6.46).

By integrating both sides of (6.45) and multiplying by i on both sides, we have $iG_1(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) = R_0(i\theta) + R_1$ with some constant $R_1 = R_1(\mathfrak{C};\mathfrak{D};\mathfrak{A})$. Repeating this operation, and by (6.46), we obtain

$$i^d G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) = \sum_{k=0}^{d} R_{d-k}\frac{(i\theta)^k}{k!}, \tag{6.47}$$

where there exist constants $R_k = R_k(\mathfrak{C};\mathfrak{D};\mathfrak{A})$ $(0 \le k \le d)$. We can explicitly determine $\{R_k\}$ as follows. Putting $\theta = \pm\pi$ in (6.47) with $d+1$ $(d \in \mathbb{N}_0)$, we have

$$\frac{i^{d+1}}{2(i\pi)}\{G_{d+1}(\pi;\mathfrak{C};\mathfrak{D};\mathfrak{A}) - G_{d+1}(-\pi;\mathfrak{C};\mathfrak{D};\mathfrak{A})\} = \sum_{\mu=0}^{[d/2]} R_{d-2\mu}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}. \tag{6.48}$$

It follows from (6.44) that the left-hand side of (6.48) is equal to

$$
\begin{aligned}
&-2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta-k)\varepsilon_{a_\eta-k}\\
&\times\sum_{\tau=0}^{[(k-1)/2]}\left\{\sum_{\omega=0}^{k-2\tau-1}\binom{d+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;k-2\tau-1-\omega;\eta)}{m^{d+\omega+1}}\right\}\frac{(i\pi)^{2\tau}}{(2\tau+1)!}.
\end{aligned}
\tag{6.49}
$$

Applying (6.41) with

$$
g(x)=\sum_{\omega=0}^{x-1}\binom{d+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;x-1-\omega;\eta)}{m^{d+\omega+1}},
$$

we can rewrite (6.48) as

$$
\sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta-1}\binom{d+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;a_\eta-1-\omega;\eta)}{m^{d+\omega+1}}=\sum_{\nu=0}^{[d/2]}R_{d-2\nu}\frac{(i\pi)^{2\nu}}{(2\nu+1)!}.
\tag{6.50}
$$

Hence, by Corollary 6.1, we have

$$
\begin{aligned}
R_d&=R_d(\mathfrak{C};\mathfrak{D};\mathfrak{A})\\
&=-2\sum_{\nu=0}^{d}\phi(d-\nu)\varepsilon_{d-\nu}\sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta-1}\binom{\nu+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;a_\eta-1-\omega;\eta)}{m^{\nu+\omega+1}}
\end{aligned}
\tag{6.51}
$$

for $d\in\mathbb{N}_0$. Therefore, combining (6.44),(6.47) and (6.51), we have

$$
\begin{aligned}
&\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^lC(l)e^{il\theta}}{l^d}-2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta-k)\varepsilon_{a_\eta-k} \qquad (6.52)\\
&\quad\times\sum_{\nu=0}^{k}\left\{\sum_{\omega=0}^{k-\nu}\binom{d-1+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^mD(m;k-\nu-\omega;\eta)e^{im\theta}}{m^{d+\omega}}\right\}\frac{(i\theta)^\nu}{\nu!}\\
&=-2\sum_{\mu=0}^{d}\sum_{\nu=0}^{d-\mu}\phi(d-\mu-\nu)\varepsilon_{d-\mu-\nu}
\end{aligned}
$$

$$\times \sum_{\eta=1}^{h} \sum_{\omega=0}^{a_\eta-1} \binom{\nu+\omega}{\omega} (-1)^\omega \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{D(m;a_\eta-1-\omega;\eta)}{m^{\nu+\omega+1}} \frac{(i\theta)^\mu}{\mu!}.$$

Changing the running indices (μ,ν) into (k,ξ) with $k=\mu+\nu$ and $\xi=\mu\leq k$, we find that the right-hand side of (6.52) is equal to the second term on the right-hand side of (6.43). □

7. Functional relations for $\zeta_3(\mathbf{s};A_3)$

In the rest of this paper, we will give explicit forms of functional relations for zeta-functions of root systems by using lemmas proved in Section 6. In this section, we consider the case of A_r type.

Fix $p\in\mathbb{N}$ and $s\in\mathbb{R}$ with $s>1$ and $x\in\mathbb{C}$ with $|x|=1$. From (4.31), we have

$$\left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} \frac{(-1)^l e^{il\theta}}{l^{2p}} - 2\sum_{j=0}^{p} \phi(2p-2j)\frac{(i\theta)^{2j}}{(2j)!}\right) \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^s} = 0 \tag{7.53}$$

for $\theta\in[-\pi,\pi]$. Hence we have

$$\sum_{\substack{l\in\mathbb{Z},\ l\neq 0\\ m\geq 1\\ l+m\neq 0}} \frac{(-1)^{l+m} x^m e^{i(l+m)\theta}}{l^{2p} m^s} - 2\sum_{j=0}^{p} \phi(2p-2j) \left\{\sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^s}\right\} \frac{(-1)^j\theta^{2j}}{(2j)!}$$
$$= -\sum_{m=1}^{\infty} \frac{x^m}{m^{s+2p}} \tag{7.54}$$

for $\theta\in[-\pi,\pi]$. Now we use Lemma 6.2 with $h=1$, $a_1=2p$,

$$C(N)=\sum_{\substack{l\neq 0,\ m\geq 1\\ l+m=N}} \frac{x^m}{l^{2p}m^s} \quad (N\in\mathbb{Z},\ N\neq 0),$$

and $D(N;\mu;1)=x^N N^{-s}$ (if $\mu=0$ and $N\geq 1$), or $=0$ (otherwise). Under these choices, we see that the left-hand side of (7.54) is of the form (6.42) because $\varepsilon_{2p-k}=1$ $(0\leq k\leq 2p)$ implies $k=2j$ $(0\leq j\leq p)$. Furthermore the right-hand side of (7.54) is a constant, because we fix s,x

and p. Therefore we can apply Lemma 6.2 with $d = 2q$ for $q \in \mathbb{N}$. Then (6.43) gives that

$$\sum_{\substack{l\neq 0,\, m\geq 1\\ l+m\neq 0}} \frac{(-1)^{l+m}x^m e^{i(l+m)\theta}}{l^{2p}m^s(l+m)^{2q}}$$
$$-2\sum_{j=0}^{p} \phi(2p-2j) \sum_{\xi=0}^{2j} \binom{2q-1+2j-\xi}{2q-1}(-1)^{2j-\xi}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+2q+2j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}$$
$$+2\sum_{j=0}^{q} \phi(2q-2j) \sum_{\xi=0}^{2j} \binom{2p-1+2j-\xi}{2p-1}(-1)^{2p-1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+2p+2j-\xi}}\frac{(i\theta)^{\xi}}{\xi!} = 0 \tag{7.55}$$

for $\theta \in [-\pi, \pi]$, where we replace k by $2j$ in (6.43) because $(a_1, d) = (2p, 2q)$ as mentioned above. This relation will play an important role in the next section. Here we apply Lemma 6.1 to the real part of (7.55) in the case $\theta = \pi$ and $x = 1$. Then we have the following.

Prop 7.1. For $p, q \in \mathbb{N}$,

$$\sum_{\substack{l\in\mathbb{Z},\ l\neq 0\\ m\geq 1\\ l+m\neq 0}} \frac{1}{l^{2p}m^s(l+m)^{2q}} = 2\sum_{\nu=0}^{p}\binom{2p+2q-2\nu-1}{2q-1}\zeta(2\nu)\zeta(s+2p+2q-2\nu)$$
$$+2\sum_{\nu=0}^{q}\binom{2p+2q-2\nu-1}{2p-1}\zeta(2\nu)\zeta(s+2p+2q-2\nu) \tag{7.56}$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides.

Note that (7.56) essentially coincides with (3.21) in the case $(k, l) = (2p, 2q)$, because the left-hand side of (7.56) can be easily transformed to that of (3.21) in the case $(k, l) = (2p, 2q)$. This implies that, from relation (7.53) which is given by multiplying two quantities of A_1 type, we can obtain relation (7.56) for zeta-functions of A_2 and A_1 type. From the view point of Dynkin diagrams, we may say that (7.53) corresponds to two vertices, and the above procedure of applying Lemma 6.2 to obtain (7.56) corresponds to the fact that the Dynkin diagram of A_2 can be produced by joining those two vertices. Based on this observation, instead of (7.53), we next combine a quantity of A_2 type and a quantity of A_1 type to get a relation for zeta-functions of A_3 and of A_2 type. From (1.5), we see that the zeta-function of root system of A_3 type is defined by

$$\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6;A_3) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(m+n)^{s_5}(l+m+n)^{s_6}}. \tag{7.57}$$

Fix $p,q,b\in\mathbb{N}$ and $s\in\mathbb{R}$ with $s>1$ and $x\in\mathbb{C}$ with $|x|=1$. From (4.31), we have

$$\left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l e^{il\theta}}{l^{2p}} - 2\sum_{j=0}^{p}\phi(2p-2j)\,\frac{(i\theta)^{2j}}{(2j)!}\right)\sum_{\substack{m\in\mathbb{Z},\, m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m+n}x^n e^{i(m+n)\theta}}{m^{2q}n^s(m+n)^{2b}} = 0 \tag{7.58}$$

for $\theta\in[-\pi,\pi]$. This formula corresponds to a diagram of A_2 and another vertex. Next we use Lemma 6.2, which gives the procedure of joining these two figures to obtain the diagram of A_3. First, by separating the terms corresponding to $l+m+n=0$, we have

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ m+n\neq 0,\ l+m+n\neq 0}}\frac{(-1)^{l+m+n}x^n e^{i(l+m+n)\theta}}{l^{2p}m^{2q}n^s(m+n)^{2b}}\\
&\quad - 2\sum_{j=0}^{p}\phi(2p-2j)\left\{\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m+n}x^n e^{i(m+n)\theta}}{m^{2q}n^s(m+n)^{2b}}\right\}\frac{(-1)^j\theta^{2j}}{(2j)!}\\
&\quad = -\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{x^n}{m^{2q}n^s(m+n)^{2p+2b}}
\end{aligned} \tag{7.59}$$

for $\theta\in[-\pi,\pi]$. We can apply Lemma 6.2 with $h=1$, $a_1=2p$,

$$C(N) = \sum_{\substack{l,m\neq 0\\ n\geq 1\\ m+n\neq 0\\ l+m+n=N}}\frac{x^n}{l^{2p}m^{2q}n^s(m+n)^{2b}},$$

$$D(N;\mu;1) = \begin{cases}\displaystyle\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n=N}}\frac{x^n}{m^{2q}n^s(m+n)^{2b}} & (\nu=0,\ N\neq 0),\\ 0 & (\nu\geq 1,\ N\neq 0),\end{cases}$$

and $d=2c$ ($c\in\mathbb{N}$). Formula (7.59) implies that the assumptions of Lemma 6.2 are satisfied, so consequently we have

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ m+n\neq 0,\ l+m+n\neq 0}} \frac{(-1)^{l+m+n}x^n e^{i(l+m+n)\theta}}{l^{2p}m^{2q}n^s(m+n)^{2b}(l+m+n)^{2c}}\\
&=2\sum_{j=0}^{p}\phi(2p-2j)\sum_{\xi=0}^{2j}\binom{2j-\xi+2c-1}{2c-1}(-1)^{2j-\xi}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m+n}x^n e^{i(m+n)\theta}}{m^{2q}n^s(m+n)^{2b+2c+2j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\
&-2\sum_{j=0}^{c}\phi(2c-2j)\sum_{\xi=0}^{2j}\binom{2j-\xi+2p-1}{2p-1}(-1)^{2p-1}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{x^n}{m^{2q}n^s(m+n)^{2p+2b+2j-\xi}}\frac{(i\theta)^\xi}{\xi!}.
\end{aligned}$$

Putting $x=-e^{-i\theta}$ ($\theta\in\mathbb{R}$) on both sides and separating the terms corresponding to $l+m=0$, we have

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ l+m\neq 0,\ m+n\neq 0\\ l+m+n\neq 0}} \frac{(-1)^{l+m} e^{i(l+m)\theta}}{l^{2p}m^{2q}n^s(m+n)^{2b}(l+m+n)^{2c}}\\
&-2\sum_{j=0}^{p}\phi(2p-2j)\sum_{\xi=0}^{2j}\binom{2j-\xi+2c-1}{2c-1}(-1)^{2j-\xi}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m} e^{im\theta}}{m^{2q}n^s(m+n)^{2b+2c+2j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\
&+2\sum_{j=0}^{c}\phi(2c-2j)\sum_{\xi=0}^{2j}\binom{2j-\xi+2p-1}{2p-1}(-1)^{2p-1}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^n e^{-in\theta}}{m^{2q}n^s(m+n)^{2p+2b+2j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\
&=-\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{1}{m^{2p+2q}n^{s+2c}(m+n)^{2b}}.
\end{aligned}$$

Again we apply Lemma 6.2 with $h=2$, $a_1=2p$, $a_2=2c$ and $d=2a$ for $a\in\mathbb{N}$. Then

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ l+m\neq 0,\ m+n\neq 0\\ l+m+n\neq 0}} \frac{(-1)^{l+m} e^{i(l+m)\theta}}{l^{2p}m^{2q}n^s(l+m)^{2a}(m+n)^{2b}(l+m+n)^{2c}}\\
&=2\sum_{j=0}^{p}\phi(2p-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2j-\xi}\binom{\omega+2a-1}{\omega}(-1)^{\omega}\binom{2j-\xi-\omega+2c-1}{2c-1}\\
&\quad\times(-1)^{2j-\xi-\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^m e^{im\theta}}{m^{2q+2a+\omega}n^s(m+n)^{2b+2c+2j-\xi-\omega}}\frac{(i\theta)^\xi}{\xi!}
\end{aligned}$$

$$-2\sum_{j=0}^{c}\phi(2c-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2j-\xi}\binom{\omega+2a-1}{\omega}(-1)^{\omega}\binom{2j-\xi-\omega+2p-1}{2p-1}$$
$$\times(-1)^{2p-1}(-1)^{2a+\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^n e^{-in\theta}}{m^{2q}n^{s+2a+\omega}(m+n)^{2p+2b+2j-\xi-\omega}}\frac{(i\theta)^{\xi}}{\xi!}$$
$$-2\sum_{j=0}^{a}\phi(2a-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2p-1}\binom{\omega+2j-\xi}{\omega}(-1)^{\omega}\binom{2p+2c-2-\omega}{2c-1}$$
$$\times(-1)^{2p-1-\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{1}{m^{2q+2j-\xi+\omega+1}n^{s}(m+n)^{2p+2b+2c-1-\omega}}\frac{(i\theta)^{\xi}}{\xi!}$$
$$+2\sum_{j=0}^{a}\phi(2a-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2c-1}\binom{\omega+2j-\xi}{\omega}(-1)^{\omega}\binom{2p+2c-2-\omega}{2p-1}$$
$$\times(-1)^{2p-1}(-1)^{2j-\xi+\omega+1}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{1}{m^{2q}n^{s+2j-\xi+\omega+1}(m+n)^{2p+2b+2c-1-\omega}}\frac{(i\theta)^{\xi}}{\xi!}$$

holds for $\theta\in[-\pi,\pi]$. Now we put $\theta=\pi$ in this equation and take its real part. For simplicity, we denote the obtained equation by $J_1=J_2+J_3+J_4+J_5$. First we consider J_1. This can be divided into the following:

$$\sum_{\substack{l\geq 1\\ m\geq 1\\ n\geq 1}}+\sum_{\substack{l\leq -1\\ m\geq 1\\ n\geq 1\\ l+m\neq 0\\ l+m+n\neq 0}}+\sum_{\substack{l\geq 1\\ m\leq -1\\ n\geq 1\\ l+m\neq 0\\ m+n\neq 0\\ l+m+n\neq 0}}+\sum_{\substack{l\leq -1\\ m\leq -1\\ n\geq 1\\ m+n\neq 0\\ l+m+n\neq 0}},$$

which we denote by $J_{11}+J_{12}+J_{13}+J_{14}$. We can immediately see that $J_{11}=\zeta_3(2p,2q,s,2a,2b,2c;A_3)$. For J_{12}, replacing l by $-l$, we have

$$J_{12}=\sum_{\substack{l\geq 1,\ m\geq 1\\ n\geq 1,\ l\neq m\\ l\neq m+n}}\frac{1}{(-l)^{2p}m^{2q}n^{s}(-l+m)^{2a}(m+n)^{2b}(-l+m+n)^{2c}}.$$

Here, putting $j=-l+m$ if $l<m$ and $k=l-m$ if $l>m$, respectively, we have

$$J_{12}=\sum_{\substack{l\geq 1,\ j\geq 1\\ n\geq 1}}\frac{1}{l^{2p}(l+j)^{2q}n^{s}j^{2a}(l+j+n)^{2b}(j+n)^{2c}}$$
$$+\sum_{\substack{k\geq 1,\ m\geq 1\\ n\geq 1,\ k\neq n}}\frac{1}{(k+m)^{2p}m^{2q}n^{s}k^{2a}(m+n)^{2b}(-k+n)^{2c}},$$

where the first term on the right-hand side is $\zeta_3(2p, 2a, s, 2q, 2c, 2b; A_3)$. Furthermore, putting $j' = -k + n$ if $k < n$ and $k' = k - n$ if $k > n$, respectively, in the second term on the right-hand side, we can obtain

$$J_{12} = \zeta_3(2p, 2a, s, 2q, 2c, 2b; A_3) + \zeta_3(2q, 2a, 2c, 2p, s, 2b; A_3) + \zeta_3(2q, s, 2c, 2b, 2a, 2p; A_3).$$

Similarly we can express J_{13} and J_{14} as sums of values of the zeta-function of A_3 type. Therefore J_1 can be transformed to the left-hand side of the following theorem. On the other hand, if we apply (6.40) to $J_2+J_3+J_4+J_5$, then it can be transformed to the right-hand side of the following theorem with

$$\mathcal{T}(2d, s, 2e) = \sum_{\substack{m\neq 0,\ n\geq 1\\ m+n\neq 0}} \frac{1}{m^{2d} n^s (m+n)^{2e}}$$

for $d, e \in \mathbb{N}$. From Proposition 7.1, we see that $\mathcal{T}(2d, s, 2e)$ can be written as (7.60) below.

Theorem 7.1. *For $p, q, a, b, c \in \mathbb{N}$,*

$$\begin{aligned}
&\zeta_3(2p, 2q, s, 2a, 2b, 2c; A_3) + \zeta_3(2p, 2a, s, 2q, 2c, 2b; A_3) + \zeta_3(2q, 2a, 2c, 2p, s, 2b; A_3)\\
&+ \zeta_3(2q, s, 2c, 2b, 2a, 2p; A_3) + \zeta_3(2a, 2p, 2b, 2q, 2c, s; A_3) + \zeta_3(2a, 2c, 2b, s, 2p, 2q; A_3)\\
&+ \zeta_3(s, 2c, 2p, 2a, 2b, 2q; A_3) + \zeta_3(2b, 2q, 2a, s, 2p, 2c; A_3) + \zeta_3(2b, s, 2a, 2q, 2c, 2p; A_3)\\
&+ \zeta_3(2p, 2b, s, 2c, 2q, 2a; A_3) + \zeta_3(2c, 2p, 2q, 2b, 2a, s; A_3) + \zeta_3(2c, 2b, 2q, 2p, s, 2a; A_3)\\
&\quad = 2\sum_{\xi=0}^{p} \zeta(2\xi) \sum_{\omega=0}^{2p-2\xi} \binom{\omega+2a-1}{\omega} \binom{2p+2c-2\xi-\omega-1}{2c-1}\\
&\qquad \times \mathcal{T}(2q+2a+\omega, s, 2p+2b+2c-2\xi-\omega)\\
&\quad + 2\sum_{\xi=0}^{c} \zeta(2\xi) \sum_{\omega=0}^{2c-2\xi} \binom{\omega+2a-1}{\omega} \binom{2p+2c-2\xi-\omega-1}{2p-1}\\
&\qquad \times \mathcal{T}(2q, s+2a+\omega, 2p+2b+2c-2\xi-\omega)\\
&\quad + 2\sum_{\xi=0}^{a} \zeta(2\xi) \sum_{\omega=0}^{2p-1} \binom{\omega+2a-2\xi}{\omega} \binom{2p+2c-2-\omega}{2c-1}\\
&\qquad \times \mathcal{T}(2q+2a-2\xi+\omega+1, s, 2p+2b+2c-1-\omega)\\
&\quad + 2\sum_{\xi=0}^{a} \zeta(2\xi) \sum_{\omega=0}^{2c-1} \binom{\omega+2a-2\xi}{\omega} \binom{2p+2c-\omega-2}{2p-1}
\end{aligned}$$

$$\times \mathcal{T}(2q, s+2a-2\xi+\omega+1, 2p+2b+2c-\omega-1)$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides, where

$$\begin{aligned}\mathcal{T}(2d, s, 2e) = 2\sum_{\nu=0}^{d} \binom{2d+2e-2\nu-1}{2e-1} \zeta(2\nu)\zeta(s+2d+2e-2\nu) \\ + 2\sum_{\nu=0}^{e} \binom{2d+2e-2\nu-1}{2d-1} \zeta(2\nu)\zeta(s+2d+2e-2\nu).\end{aligned} \tag{7.60}$$

Example 7.1. In the case when $(p, q, a, b, c) = (k, k, k, k, k)$ and $s = 2k$ for $k \in \mathbb{N}$ in Theorem 7.1, we recover the explicit expression for Witten's volume formula of A_3 type, which has been proved by Gunnells and Sczech ([10, Proposition 8.5]). For example, in the case when $(p, q, a, b, c) = (1, 1, 1, 1, 1)$, we obtain

$$\begin{aligned}&4\zeta_3(2, 2, s, 2, 2, 2; A_3) + 2\zeta_3(2, s, 2, 2, 2, 2; A_3) \\ &\quad + 4\zeta_3(2, 2, 2, s, 2, 2; A_3) + 2\zeta_3(2, 2, 2, 2, 2, s; A_3) \\ &= 678\zeta(s+10) - 512\zeta(2)\zeta(s+8) + 148\zeta(4)\zeta(s+6) + 4\zeta(6)\zeta(s+4),\end{aligned} \tag{7.61}$$

because $\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6; A_3) = \zeta_3(s_3, s_2, s_1, s_5, s_4, s_6; A_3)$. In particular when $s = 2$, we obtain the explicit value of $C_W(2, A_3)$, that is,

$$\zeta_3(2, 2, 2, 2, 2, 2; A_3) = \frac{23}{2554051500}\pi^{12}. \tag{7.62}$$

In our previous work [16, Theorem 3.4], we already obtained the functional relation between $\zeta_3(\mathbf{s}; A_3)$ and $\zeta_2(\mathbf{s}; A_2)$, and checked that the functional relation implicitly implies (7.62), by using the properties of $\zeta_2(\mathbf{s}; A_2)$. On the other hand, we can see that the above formula in Theorem 7.1 itself includes the explicit form of Witten's volume formula of A_3 type.

Example 7.2. By the same method as above, we can obtain the following formulas ([30]):

$$\begin{aligned}&\zeta_3(1, 1, 1, 2, 1, 2; A_3) = -\frac{29}{175}\zeta(2)^4 + \zeta(3)\zeta(5) - \frac{1}{2}\zeta(6, 2), \\ &\zeta_3(1, 1, 2, 1, 2, 1; A_3) = \frac{2683}{1050}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 16\zeta(3)\zeta(5) + \frac{29}{4}\zeta(6, 2), \\ &\zeta_3(1, 1, 1, 2, 1, 3; A_3) = \frac{2}{5}\zeta(2)^2\zeta(5) + 10\zeta(2)\zeta(7) - \frac{53}{3}\zeta(9).\end{aligned}$$

Remark 7.1. Here we summarize the method developed in this section. The starting point is the simple identity (4.31) (and (4.32)), which is based on the fact $\zeta(-2n) = 0$ $(n \in \mathbb{N})$. One basic idea is to multiply (4.31) by an infinite series (see (7.53))) to obtain a new identity (see (7.54)). Then we apply the argument of repeated integration, embodied in Lemma 6.2, to deduce the functional relations. This procedure is the essence of the "u-method" mentioned in Sections 2 and 3, though the parameter $u > 1$ does not appear in this section.

However, the original u-method (developed, for instance, in [30]) is unsatisfactory because it only produces functional relations in which some of the variables should be equal to 0. In order to remove this restriction, we introduce the idea of considering the infinite series of polylogarithm type (that is, with an additional parameter x in the numerators). This idea, inspired by the method in [32] (see Remark 3.3), was first successfully used in [16] under the name of the "polylogarithm technique". This additional flexibility enables us to deduce more general type of functional relations such as Theorem 7.1. We will also use this technique in the following sections.

We may proceed further. Next we combine a quantity of A_3 type and a quantity of A_1 type to obtain

$$\left(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{(-1)^k e^{ik\theta}}{k^{2p}} - 2\sum_{j=0}^{p} \phi(2p-2j)\,\frac{(i\theta)^{2j}}{(2j)!}\right) \times \sum_{\substack{l,m\in\mathbb{Z},\, n\geq 1,\\ l,m\neq 0,\, l+m\neq 0\\ m+n\neq 0,\, l+m+n\neq 0}} \frac{(-1)^{l+m+n} x^m y^n e^{i(l+m+n)\theta}}{l^{2q} m^{2r} n^s (l+m)^{2a} (m+n)^{2b} (l+m+n)^{2c}} = 0$$

for $p, q, r, a, b, c \in \mathbb{N}$ and $x, y \in \mathbb{C}$ with $|x| = 1$ and $|y| = 1$. Again, by using Lemma 6.2 repeatedly, we will be able to obtain the functional relation for zeta-functions of A_4 and A_3 type. Then, by using the result in Theorem 7.1, we will be able to obtain functional relations for zeta-functions of A_4 and A_1 type, which include explicit forms of Witten's volume formulas of A_4 type, for example,

$$\zeta_4(2,2,2,2,2,2,2,2,2,2; A_4) = \frac{1}{650970015609375}\pi^{20}. \tag{7.63}$$

By continuing this procedure inductively, it seems to be possible to obtain functional relations which include explicit forms of Witten's volume formulas of A_r type for any $r \in \mathbb{N}$.

8. Functional relations for $\zeta_2(\mathbf{s}; C_2)$

In this section, we study

$$\zeta_2(s_1, s_2, s_3, s_4; C_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}} \tag{8.64}$$

(see [18, (6.1)], also [19, Example 7.3]). As noted in [18, Section 2], we know that

$$\zeta_2(s_1, s_2, s_3, s_4; B_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (2m+n)^{s_4}} \tag{8.65}$$

(see [18, (2.11)]), which coincides with $\zeta_2(s_2, s_1, s_3, s_4; C_2)$. This fact is the natural consequence of the isomorphism $B_2 \simeq C_2$.

Here we consider $\zeta_2(\mathbf{s}; C_2)$ and construct explicit functional relations which include explicit forms of Witten's volume formulas of C_2 type.

As we mentioned in the previous section, the procedure of producing a functional relation for $\zeta_2(\mathbf{s}; A_2)$ corresponds to the fact that the Dynkin diagram of A_2 can be produced by adding one edge which joins two vertices. From this viewpoint, we should step on the procedure corresponding to adding another edge to the Dynkin diagram of A_2 to obtain the diagram of C_2, by using Lemma 6.2.

Replacing x by $-xe^{i\theta}$ on the left-hand side of (7.55), we have

$$\begin{aligned} &\sum_{\substack{l\in\mathbb{Z},\, l\neq 0 \\ m\geq 1,\, l+m\neq 0}} \frac{(-1)^l x^m e^{i(l+2m)\theta}}{l^{2p} m^s (l+m)^{2q}} \\ &- 2\sum_{j=0}^{p} \phi(2p-2j) \sum_{\xi=0}^{2j} \binom{2q-1+2j-\xi}{2q-1} (-1)^{2j-\xi} \sum_{m=1}^{\infty} \frac{x^m e^{2im\theta}}{m^{s+2q+2j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\ &- 2\sum_{j=0}^{q} \phi(2q-2j) \sum_{\xi=0}^{2j} \binom{2p-1+2j-\xi}{2p-1} \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^{s+2p+2j-\xi}} \frac{(i\theta)^\xi}{\xi!} = 0 \end{aligned}$$

for $\theta \in [-\pi, \pi]$. From the first sum, we separate the terms corresponding to the condition $l + 2m = 0$ and move them to the right-hand side. Then, as well as in the case of (7.54), applying Lemma 6.2 with $(h, a_1, a_2, d) =$

$(2,2p,2q,2r)$ for $r\in\mathbb{N}$, we obtain

$$\sum_{\substack{l\in\mathbb{Z},\,l\neq 0\\ m\geq 1,\,l+m\neq 0\\ l+2m\neq 0}} \frac{(-1)^l x^m e^{i(l+2m)\theta}}{l^{2p}m^s(l+m)^{2q}(l+2m)^{2r}} \tag{8.66}$$

$$\begin{aligned}
&-2\sum_{j=0}^{p}\phi(2p-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2j-\xi}\binom{\omega+2r-1}{\omega}(-1)^{\omega}\\
&\qquad\times\binom{2q-1+2j-\xi-\omega}{2q-1}(-1)^{2j-\xi-\omega}\frac{1}{2^{2r+\omega}}\sum_{m=1}^{\infty}\frac{x^m e^{2im\theta}}{m^{s+2q+2j-\xi+2r}}\frac{(i\theta)^{\xi}}{\xi!}\\
&-2\sum_{j=0}^{q}\phi(2q-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2j-\xi}\binom{\omega+2r-1}{\omega}(-1)^{\omega}\\
&\qquad\times\binom{2p-1+2j-\xi-\omega}{2p-1}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+2p+2j-\xi+2r}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+2\sum_{j=0}^{r}\phi(2r-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2p-1}\binom{\omega+2j-\xi}{\omega}(-1)^{\omega}\\
&\qquad\times\binom{2p+2q-2-\omega}{2q-1}(-1)^{2p-1-\omega}\frac{1}{2^{2j-\xi+\omega+1}}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+2q+2j-\xi+2p}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+2\sum_{j=0}^{r}\phi(2r-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2q-1}\binom{\omega+2j-\xi}{\omega}(-1)^{\omega}\\
&\qquad\times\binom{2p+2q-2-\omega}{2p-1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+2p+2j-\xi+2q}}\frac{(i\theta)^{\xi}}{\xi!}=0
\end{aligned}$$

for $\theta\in[-\pi,\pi]$. Then, putting $(x,\theta)=(1,\pi)$ in (8.66) and applying Lemma 6.1 to the real part of this equation, we obtain the following relation which holds for $s>1$, and furthermore for $s\in\mathbb{C}$ except for singularities by the meromorphic continuation of $\zeta_2(\mathbf{s};C_2)$.

Theorem 8.1. *For $p,q,r\in\mathbb{N}$,*

$$\begin{aligned}
&\zeta_2(2p,s,2q,2r;C_2)+\zeta_2(2p,2q,s,2r;C_2)+\zeta_2(2r,2q,s,2p;C_2)+\zeta_2(2r,s,2q,2p;C_2)\\
&=2\sum_{\nu=0}^{p}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s)\sum_{\mu=0}^{2p-2\nu}\frac{1}{2^{2r+\mu}}\binom{2p+2q-2\nu-\mu-1}{2q-1}\binom{2r-1+\mu}{2r-1}\\
&+2\sum_{\nu=0}^{q}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s)\sum_{\mu=0}^{2q-2\nu}(-1)^{\mu}\binom{2p+2q-2\nu-\mu-1}{2p-1}\binom{2r-1+\mu}{2r-1}
\end{aligned}$$

$$+2\sum_{\nu=0}^{r}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s)\sum_{\mu=0}^{2p-1}\frac{1}{2^{2r-2\nu+\mu+1}}\binom{2p+2q-\mu-2}{2q-1}\binom{2r-2\nu+\mu}{2r-2\nu}$$
$$+2\sum_{\nu=0}^{r}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s)\sum_{\mu=0}^{2q-1}(-1)^{\mu+1}\binom{2p+2q-\mu-2}{2p-1}\binom{2r-2\nu+\mu}{2r-2\nu}$$

holds for all $s\in\mathbb{C}$ except for singularities of functions on both sides. Note that singularities of $\zeta_2(\mathbf{s},C_2)$ have been determined in [18, Theorem 6.2].

Example 8.1. Putting $(p,q,r)=(1,1,1)$ in Theorem 8.1, we can obtain

$$\zeta_2(2,s,2,2;C_2)+\zeta_2(2,2,s,2;C_2)=-\frac{39}{16}\zeta(s+6)+\frac{3}{2}\zeta(2)\zeta(s+4). \quad (8.67)$$

In particular when $s=2$, we obtain

$$\zeta_2(2,2,2,2;C_2)=\frac{\pi^8}{302400}, \quad (8.68)$$

which have already been obtained in [19, (7.24)]. It should be noted that Equations (8.67) and (8.70) mentioned below coincide with Equations (2.6) and (2.7) in [16], respectively. Note that, in [16], we used the notation $\zeta_2(s_1,s_2,s_3,s_4;B_2)$ defined by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m+n)^{s_1}n^{s_2}m^{s_3}(m+n)^{s_4}},$$

different from (8.65) (see [18, Section 2]).

Remark 8.1. Using the same method as in the proof of Theorem 8.1, we can prove that

$$\zeta_2(p,s,q,r;C_2)+(-1)^p\zeta_2(p,q,s,r;C_2)+(-1)^{p+q}\zeta_2(r,q,s,p;C_2) \quad (8.69)$$
$$+(-1)^{p+q+r}\zeta_2(r,s,q,p;C_2)$$
$$=2(-1)^p\times$$
$$\left\{\sum_{\nu=0}^{[p/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\sum_{\mu=0}^{p-2\nu}\frac{1}{2^{r+\mu}}\binom{p+q-2\nu-\mu-1}{q-1}\binom{r-1+\mu}{r-1}\right.$$
$$+\sum_{\nu=0}^{[q/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\sum_{\mu=0}^{q-2\nu}(-1)^{\mu}\binom{p+q-2\nu-\mu-1}{p-1}\binom{r-1+\mu}{r-1}$$

$$+\sum_{\nu=0}^{[r/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\sum_{\mu=0}^{p-1}\frac{1}{2^{r-2\nu+\mu+1}}\binom{p+q-\mu-2}{q-1}\binom{r-2\nu+\mu}{r-2\nu}$$
$$+\sum_{\nu=0}^{[r/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\sum_{\mu=0}^{q-1}(-1)^{\mu+1}\binom{p+q-\mu-2}{p-1}\binom{r-2\nu+\mu}{r-2\nu}\Bigg\}$$

holds for all $s\in\mathbb{C}$ except for singularities of functions on both sides, where $p,q,r\in\mathbb{N}$. For example, we have

$$\zeta_2(2,s,2,1;C_2)+\zeta_2(2,2,s,1;C_2)+\zeta_2(1,2,s,2;C_2)-\zeta_2(1,s,2,2;C_2) \tag{8.70}$$
$$=3\zeta(2)\zeta(s+3)-\frac{39}{8}\zeta(s+5).$$

In particular, putting $s=2$ in (8.70), we have

$$\zeta_2(2,2,2,1;C_2)=\frac{3}{2}\zeta(2)\zeta(5)-\frac{39}{16}\zeta(7),$$

which coincides with our previous result in [44, Example in §3]. Note that the left-hand side of (8.69) is equal to $S(\mathbf{s},\mathbf{y};I;\Delta)$ for $\Delta=\Delta(C_2)$, $\mathbf{s}=(p,s,q,r)$, $\mathbf{y}=0$ and $I=\{2\}$. Therefore we can see that Theorem 8.1 corresponds to the case C_2 of Theorem 5.1.

9. Functional relations for $\zeta_3(\mathbf{s};B_3)$ and for $\zeta_3(\mathbf{s};C_3)$

In this section, we consider $\zeta_3(\mathbf{s};B_3)$ and $\zeta_3(\mathbf{s};C_3)$ defined by

$$\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8,s_9;B_3)$$
$$=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}m_1^{-s_1}m_2^{-s_2}m_3^{-s_3}(m_1+m_2)^{-s_4}(m_2+m_3)^{-s_5}(2m_2+m_3)^{-s_6}$$
$$\times(m_1+m_2+m_3)^{-s_7}(m_1+2m_2+m_3)^{-s_8}(2m_1+2m_2+m_3)^{-s_9}, \tag{9.71}$$

and

$$\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8,s_9;C_3)$$
$$=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty}m_1^{-s_1}m_2^{-s_2}m_3^{-s_3}(m_1+m_2)^{-s_4}(m_2+m_3)^{-s_5}(m_2+2m_3)^{-s_6}$$
$$\times(m_1+m_2+m_3)^{-s_7}(m_1+m_2+2m_3)^{-s_8}(m_1+2m_2+2m_3)^{-s_9}, \tag{9.72}$$

which have been continued meromorphically to the whole space whose possible singularities have been determined in [18, Theorems 6.1 and 6.3]. Note that $\zeta_3(\mathbf{s}; D_3)$ essentially coincides with $\zeta_3(\mathbf{s}; A_3)$ which has been considered in [16,30].

We aim to prove functional relations for these functions, namely generalize the result in Theorems 7.1 and 8.1 to the cases B_3 and C_3. However, it seems too complicated to treat these cases in full generality. Hence we study some special cases as follows.

First we prove the following functional relation for $\zeta_3(\mathbf{s}; C_3)$. The basic structure of the proof, based on Lemma 6.2, is similar to that in the proof of Theorem 7.1 for $\zeta_3(\mathbf{s}; A_3)$. A novel point here is that we will also use the result described in Section 4.

Theorem 9.1. *The functional relation*

$$\begin{aligned} &8\zeta_3(2,2,s,2,2,2,2,2,2;C_3)+8\zeta_3(2,2,2,2,s,2,2,2,2;C_3)\\ &\quad+8\zeta_3(2,2,2,2,2,2,s,2,2;C_3)\\ &=\frac{184775}{512}\zeta(s+16)-\frac{16875}{64}\zeta(2)\zeta(s+14)+\frac{513}{8}\zeta(4)\zeta(s+12)\\ &\quad+\frac{25}{8}\zeta(6)\zeta(s+10)+\frac{1}{4}\zeta(8)\zeta(s+8) \end{aligned} \tag{9.73}$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides. In particular when $s = 2$,

$$\zeta_3(2,2,2,2,2,2,2,2,2;C_3)=\frac{19}{8403115488768000}\pi^{18}, \tag{9.74}$$

hence $C_W(2, C_3) = 19/16209713520$ in Witten's volume formula (1.4).

Proof. Instead of (7.53) or (7.58), we start the same argument as in the proof of Proposition 7.1 or Theorem 7.1 from the relation

$$\{G(\theta;2,2,2;x)+G(-\theta;2,2,2;x^{-1})\}\sum_{n=1}^{\infty}\frac{(-1)^n y^n e^{in\theta}}{n^s}=0 \tag{9.75}$$

for $s > 1$, where we denote by $G(\theta; 2, 2, 2; x)$ the left-hand side of (7.55) in the case $(2p, 2q, s) = (2, 2, 2)$. Then, by replacing $-m$ ($m \geq 1$) by m

$(m \le -1)$ on the left-hand side of (9.75), we can rewrite (9.75) to

$$\sum_{\substack{l\neq 0,\, m\neq 0\\ n\geq 1,\, l+m\neq 0}} \frac{(-1)^{l+m+n} x^m y^n e^{i(l+m+n)\theta}}{l^2 m^2 n^s (l+m)^2}$$
$$-2\sum_{j=0}^{1} \phi(2-2j) \sum_{\xi=0}^{2j} \binom{1+2j-\xi}{1} (-1)^{2j-\xi} \sum_{\substack{m\neq 0\\ n\geq 1}} \frac{(-1)^{m+n} x^m y^n e^{i(m+n)\theta}}{m^{4+2j-\xi} n^s} \frac{(i\theta)^\xi}{\xi!}$$
$$+2\sum_{j=0}^{1} \phi(2-2j) \sum_{\xi=0}^{2j} \binom{1+2j-\xi}{1} (-1) \sum_{\substack{m\neq 0\\ n\geq 1}} \frac{(-1)^{n} x^m y^n e^{in\theta}}{m^{4+2j-\xi} n^s} \frac{(i\theta)^\xi}{\xi!} = 0.$$

As well as in the proof of Theorem 7.1, separate the constant terms corresponding to $l+m+n=0$ in the first term and to $m+n=0$ in the second term on the left-hand side, move them to the right-hand side, and apply Lemma 6.2 with $d=2$. Then we obtain

$$\sum_{\substack{l\neq 0,\, m\neq 0\\ n\geq 1,\, l+m\neq 0\\ l+m+n\neq 0}} \frac{(-1)^{l+m+n} x^m y^n e^{i(l+m+n)\theta}}{l^2 m^2 n^s (l+m)^2 (l+m+n)^2}$$
$$-2\sum_{j=0}^{1} \phi(2-2j) \sum_{\xi=0}^{2j} \sum_{\omega=0}^{2j-\xi} \binom{\omega+1}{\omega} (-1)^\omega \binom{1+2j-\xi-\omega}{1} (-1)^{2j-\xi-\omega}$$
$$\times \sum_{\substack{m\neq 0\\ n\geq 1}} \frac{(-1)^{m+n} x^m y^n e^{i(m+n)\theta}}{m^{4+2j-\xi-\omega} n^s (m+n)^{2+\omega}} \frac{(i\theta)^\xi}{\xi!}$$
$$+\cdots = 0,$$

where we omit three terms on the left-hand side, which are of the form similar to the second term on the left-hand side. Note that each of their denominators is of the form of A_2 type. Next we replace y by $-ye^{i\theta}$, move the constant terms to the right-hand side and apply Lemma 6.2 with $d=2$.

Then we have

$$\sum_{\substack{l\neq 0,\, m\neq 0\\ n\geq 1,\, l+m\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0}} \frac{(-1)^{l+m}x^m y^n e^{i(l+m+2n)\theta}}{l^2m^2n^s(l+m)^2(l+m+n)^2(l+m+2n)^2}$$

$$-2\sum_{j=0}^{1}\phi(2-2j)\sum_{\xi=0}^{2j}\sum_{\sigma=0}^{2j-\sigma}\binom{\sigma+1}{\sigma}(-1)^{\sigma}\sum_{\omega=0}^{2j-\xi-\sigma}\binom{\omega+1}{\omega}(-1)^{2j-\xi-\sigma}$$

$$\times\binom{1+2j-\xi-\sigma-\omega}{1}\sum_{\substack{m\neq 0\\ n\geq 1}}\frac{(-1)^m x^m y^n e^{i(m+2n)\theta}}{m^{4+2j-\xi-\omega}n^s(m+n)^{2+\omega}(m+2n)^{2+\sigma}}\frac{(i\theta)^{\xi}}{\xi!}$$

$$+\cdots=0,$$

where we omit seven terms of the forms similar to the second term. Each denominator of these terms is of the form of C_2 type. Replacing x by $-xe^{i\theta}$, applying Lemma 6.2 with $d=2$, and putting $\theta=\pi$, we obtain

$$\sum_{\substack{l\neq 0,\, m\neq 0\\ n\geq 1,\, l+m\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} \frac{x^m y^n}{l^2m^2n^s(l+m)^2(l+m+n)^2(l+m+2n)^2(l+2m+2n)^2}+\cdots=0, \tag{9.76}$$

where the omitted terms are of the form of C_2 type. Next, we replace (x,y) by $\left(xe^{i\theta},ye^{i\theta}\right)$ and $\left(xe^{-i\theta},ye^{-i\theta}\right)$ respectively, and subtract these terms. Then we have

$$\sum_{\substack{l\neq 0,\, m\neq 0\\ n\geq 1,\, l+m\neq 0\\ m+n\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} \frac{x^m y^n \sin((m+n)\theta)}{l^2m^2n^s(l+m)^2(l+m+n)^2(l+m+2n)^2(l+2m+2n)^2}+\cdots=0, \tag{9.77}$$

where the omitted terms are double series of similar forms.

Finally, in order to complete the proof of this theorem, we need to apply Theorem 4.1 to each term on the left-hand side of (9.77) with $s=2$. Then we consequently obtain

$$\sum_{\substack{l\neq 0,\, m\neq 0\\ n\geq 1,\, l+m\neq 0\\ m+n\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} \frac{x^m y^n}{l^2m^2n^s(l+m)^2(m+n)^2(l+m+n)^2(l+m+2n)^2(l+2m+2n)^2}$$

$$+\cdots=0. \tag{9.78}$$

We further replace (x, y) by $\left(e^{i\theta}, e^{2i\theta}\right)$ and $\left(e^{-i\theta}, e^{-2i\theta}\right)$ respectively, subtract these terms, and apply Theorem 4.1 with $s = 2$. Then we obtain

$$\sum_{\substack{l\neq 0,\ m\neq 0\\ n\geq 1,\ l+m\neq 0\\ m+n\neq 0,\ m+2n\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} \frac{1}{l^2m^2n^s(l+m)^2(m+n)^2(m+2n)^2(l+m+n)^2(l+m+2n)^2(l+2m+2n)^2} + \cdots = 0, \tag{9.79}$$

where the omitted terms are finite sums of zeta values of C_2 type. Though we omit their explicit forms, we can also apply Theorem 8.1 to these terms, and can express them as the right-hand side of (9.73). On the other hand, similarly to the case of Theorem 7.1, we can transform the first term on the left-hand side of (9.79) to the left-hand side of (9.73).

Moreover, from (2.16) in [18], we can easily check that $K(C_3) = 720$ (see definition (1.7)). Hence, combining (1.6) and (9.74), we obtain the value of $C_W(2, C_3)$. □

Similarly we can obtain the following formula in the case B_3.

Theorem 9.2. *The functional relation*

$$\begin{aligned}
&4\zeta_3(2,s,2,2,2,2,2,2,2;B_3) + 4\zeta_3(s,2,2,2,2,2,2,2,2;B_3)\\
&\ + 4\zeta_3(2,2,2,s,2,2,2,2,2;B_3) + 4\zeta_3(2,2,2,2,s,2,2,2,2;B_3)\\
&\ + 4\zeta_3(2,2,2,2,2,2,s,2,2;B_3) + 4\zeta_3(2,2,2,2,2,2,2,s,2;B_3)\\
&= \left(9\cdot 2^{-s-6} + \frac{5626955}{256}\right)\zeta(s+16) + \left(5\cdot 2^{-s-5} - \frac{59131}{4}\right)\zeta(2)\zeta(s+14)\\
&\quad + \left(5\cdot 2^{-s-5} + \frac{17155}{8}\right)\zeta(4)\zeta(s+12) + \frac{241}{16}\zeta(6)\zeta(s+10) + \frac{1}{8}\zeta(8)\zeta(s+8)
\end{aligned} \tag{9.80}$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides. In particular when $s = 2$, we have

$$\zeta_3(2,2,2,2,2,2,2,2,2;B_3) = \frac{19}{8403115488768000}\pi^{18}, \tag{9.81}$$

hence $C_W(2, B_3) = 19/16209713520$ in Witten's volume formula (1.4).

Proof. The argument is similar to that in the proof of Theorem 9.1. In fact, instead of (9.75), we start the same argument from the relation

$$\{H(\theta; 2,2,2,2;x) + H(-\theta; 2,2,2,2;x)\} \sum_{n=1}^{\infty} \frac{(-1)^n y^n e^{in\theta}}{n^s} = 0 \qquad (9.82)$$

for $s > 1$, where we denote by $H(\theta; 2,2,2,2;x)$ the left-hand side of (8.66) in the case $(2p, 2q, 2r, s) = (2,2,2,2)$. Repeating the same procedure as in the proof of Theorem 9.1, we can describe the left-hand side of (9.80) as a finite sum of the forms of the left-hand side of (8.69). Hence, by using (8.69), we can obtain (9.80). The value of $C_W(2, B_3)$ can be calculated from (1.6), (9.81), and the fact $K(B_3) = 720$. □

Remark 9.1. Comparing the above two theorems, we see that $C_W(2, B_3) = C_W(2, C_3)$. However it does not always hold that $C_W(2k, B_3) = C_W(2k, C_3)$, that is, $\zeta_W(2k; B_3) = \zeta_W(2k; C_3)$ for $k \geq 2$. In fact, we can compute that

$$\zeta_W(4; B_3) = 1.00066856607695295\cdots,$$
$$\zeta_W(4; C_3) = 1.00082905650461486\cdots,$$

hence $C_W(4, B_3) \neq C_W(4, C_3)$.

Note that the left-hand side of (9.73) corresponds to $S(\mathbf{s}, \mathbf{y}; I; \Delta)$ for $\Delta = \Delta(C_3)$, $\mathbf{s} = (2, s, 2, 2, 2, 2, 2, 2, 2)$, $\mathbf{y} = 0$ and $I = \{3\}$ in the terminology of Section 5. Next we prove the following result which is corresponding to the case $\Delta = \Delta(C_3)$, $\mathbf{s} = (2, s, t, 2, u, v, 2, 2, 2)$, $\mathbf{y} = 0$ and $I = \{2, 3\}$.

Theorem 9.3. *The functional relation*

$$\begin{aligned}
&\zeta_3(2, s, t, 2, u, v, 2, 2, 2; C_3) + 2\zeta_3(2, 2, t, s, 2, 2, u, v, 2; C_3) \\
&\qquad + 2\zeta_3(s, 2, 2, 2, t, 2, u, 2, v; C_3) \\
&= \sum_{\xi=0,1} \zeta(2\xi) \sum_{\tau=0}^{2-2\xi} (\tau+1) \sum_{\nu=0}^{2-2\xi-\tau} (\nu+1) \\
&\quad \times \Bigg\{ \sum_{\omega=0}^{2-2\xi-\tau-\nu} (\omega+1)(3-2\xi-\tau-\nu-\omega) \\
&\qquad \times \frac{1}{2^{\tau+2}} \zeta_2(s+2+\omega, t, u+6-2\xi-\nu-\omega, v+2+\nu; C_2)
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{\tau+\nu}\sum_{\omega=0}^{2-2\xi-\tau-\nu}(\omega+1)(3-2\xi-\tau-\nu-\omega)\\
&\quad\times\zeta_2(s,t+4+\omega+\nu,u+6-2\xi-\omega-\nu,v;C_2)\\
&+(-1)^{\tau+\nu}\sum_{\omega=0,1}\binom{\omega+2-2\xi-\tau-\nu}{\omega}(2-\omega)\\
&\quad\times\frac{1}{2^{\nu+2}}\,\zeta_2(s+3-2\xi-\tau-\nu+\omega,t+2+\nu,u+3-\omega,v+2+\tau;C_2)\\
&+\sum_{\omega=0,1}\binom{\omega+2-2\xi-\tau-\nu}{\omega}(2-\omega)\\
&\quad\times\frac{1}{2^{\nu+2}}\,\zeta_2(s,t+5-2\xi-\tau+\omega,u+3-\omega,v+2+\tau;C_2)\Big\}\\
-&\sum_{\xi=0,1}\zeta(2\xi)\sum_{\tau=0}^{2-2\xi}(\tau+1)\sum_{\nu=0,1}\binom{\nu+2-2\xi-\tau}{\nu}\\
&\times\Big\{(-1)^{\tau+1}\sum_{\omega=0}^{1-\nu}(\omega+1)(2-\nu-\omega)\\
&\quad\times\zeta_2(q,r+3-\nu-\omega,v+3-2\xi-\tau+\nu,p+4+\tau+\omega;C_2)\\
&+(-1)^{\tau+\nu}\sum_{\omega=0}^{1-\nu}(\omega+1)(2-\nu-\omega)\\
&\quad\times\zeta_2(s+2+\tau,t+5-2\xi-\tau+\nu+\omega,u+3-\nu-\omega,v;C_2)\\
&+(-1)^{\tau+\nu}\sum_{\omega=0,1}\binom{\omega+1-\nu}{\omega}(2-\omega)\\
&\quad\times\frac{1}{2^{3-2\xi-\tau+\nu}}\,\zeta_2(s+4+\tau-\nu+\omega,t+3-2\xi-\tau+\nu,u+3-\omega,v;C_2)\\
&+(-1)^{\tau+1}\sum_{\omega=0,1}\binom{\omega+1-\nu}{\omega}(2-\omega)\\
&\quad\times\frac{1}{2^{3-2\xi-\tau+\nu}}\,\zeta_2(s+2+\tau,t+5-2\xi-\tau+\omega,u+3-\omega,v;C_2)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
-&\sum_{\xi=0,1}\zeta(2\xi)\sum_{\tau=0,1}\binom{\tau+2-2\xi}{\tau}\sum_{\nu=0}^{1-\tau}(\nu+1)\\
&\times\Big\{-\sum_{\omega=0}^{1-\tau-\nu}(\omega+1)(2-\tau-\nu-\omega)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \frac{1}{2^{3-2\xi+\tau}}\zeta_2(s+2+\omega,t,u+6-2\xi-\nu-\omega,v+2+\nu,;C_2)\\
&\quad + (-1)^{\tau+\nu}\sum_{\omega=0}^{1-\tau-\nu}(\omega+1)(2-\tau-\nu-\omega)\\
&\quad \times \zeta_2(s,t+4+\nu+\omega,u+6-2\xi-\nu-\omega,v;C_2)\\
&\quad + (-1)^{\tau+\nu}\sum_{\omega=0,1}\binom{\omega+1-\tau-\nu}{\omega}(2-\omega)\\
&\quad \times \frac{1}{2^{\nu+2}}\zeta_2(s+2-\tau-\nu+\omega,t+2+\nu,u+3-\omega,v+3-2\xi+\tau;C_2)\\
&\quad - \sum_{\omega=0,1}\binom{\omega+1-\tau-\nu}{\omega}(2-\omega)\\
&\quad \times \frac{1}{2^{\nu+2}}\zeta_2(s,t+4-\tau+\omega,u+3-\omega,v+3-2\xi+\tau;C_2)\Big\}\\
&+ \sum_{\xi=0,1}\zeta(2\xi)\sum_{\tau=0,1}\binom{\tau+2-2\xi}{\tau}\sum_{\nu=0,1}\binom{\nu+1-\tau}{\nu}\\
&\quad \times\Big\{(-1)^{\tau+1}\sum_{\omega=0}^{1-\nu}(\omega+1)(2-\nu-\omega)\\
&\quad \times \zeta_2(s+5-2\xi+\tau+\omega,t,u+3-\nu-\omega,v+2-\tau+\nu;C_2)\\
&\quad + (-1)^{\tau+\nu}\sum_{\omega=0}^{1-\nu}(\omega+1)(2-\nu-\omega)\\
&\quad \times \zeta_2(s+3-2\xi+\tau,t+4+\omega-\tau+\nu,u+3-\nu-\omega,v;C_2)\\
&\quad + (-1)^{\tau+\nu}\sum_{\omega=0,1}\binom{\omega+1-\nu}{\omega}(2-\omega)\\
&\quad \times \frac{1}{2^{\nu+2-\tau}}\zeta_2(s+5-2\xi+\tau-\nu+\omega,t+2-\tau+\nu,u+3-\omega,v;C_2)\\
&\quad + (-1)^{\tau+1}\sum_{\omega=0,1}\binom{\omega+1-\nu}{\omega}(2-\omega)\\
&\quad \times \frac{1}{2^{\nu+2-\tau}}\zeta_2(s+3-2\xi+\tau,t+4-\tau+\omega,u+3-\omega,v;C_2)\Big\}
\end{aligned}
$$

holds for $s,t,u,v\in\mathbb{C}$ *except for singularities of functions on both sides.*

Proof. As well as (7.53) and (7.58), we begin by combining quantities of type A_1 and of type C_2, that is,

$$\left(\sum_{l=1}^{\infty}\frac{(-1)^l(e^{il\theta}+e^{-il\theta})}{l^2}-2\sum_{j=0,1}\ \phi(2-2j)\ \frac{(i\theta)^{2j}}{(2j)!}\right) \times\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^my^ne^{i(m+n)\theta}}{m^sn^t(m+n)^u(m+2n)^v}=0 \tag{9.83}$$

for $\theta\in[-\pi,\pi]$, where we fix $s,t,u,v\in\{z\in\mathbb{R}\,|\,z>1\}$ and $x,y\in\{z\in\mathbb{R}\,|\,|z|=1\}$. Then

$$\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m+n}x^my^ne^{i(l+m+n)\theta}}{l^2m^sn^t(m+n)^u(m+2n)^v}+\sum_{\substack{l,m,n=1\\ l\neq m+n}}^{\infty}\frac{(-1)^{l+m+n}x^my^ne^{i(-l+m+n)\theta}}{l^2m^sn^t(m+n)^u(m+2n)^v} -2\sum_{j=0,1}\ \phi(2-2j)\ \frac{(-1)^j\theta^{2j}}{(2j)!}\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^my^ne^{i(m+n)\theta}}{m^sn^t(m+n)^u(m+2n)^v} =-\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^sn^t(m+n)^{u+2}(m+2n)^v} \tag{9.84}$$

for $\theta\in[-\pi,\pi]$. Applying Lemma 6.2 with $d=2$, we have

$$\begin{aligned}&\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m+n}x^my^ne^{i(l+m+n)\theta}}{l^2m^sn^t(m+n)^u(m+2n)^v(l+m+n)^2}\\ &+\sum_{\substack{l,m,n=1\\ l\neq m+n}}^{\infty}\frac{(-1)^{l+m+n}x^my^ne^{i(-l+m+n)\theta}}{l^2m^sn^t(m+n)^u(m+2n)^v(-l+m+n)^2}\\ &-2\sum_{j=0,1}\ \phi(2-2j)\ \sum_{\xi=0}^{2j}\binom{2j-\xi+1}{2j-\xi}(-1)^{2j-\xi}\frac{(i\theta)^\xi}{\xi!}\\ &\qquad\times\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^my^ne^{i(m+n)\theta}}{m^sn^t(m+n)^{u+2+2j-\xi}(m+2n)^v}\\ &-2\sum_{j=0,1}\ \phi(2-2j)\ \sum_{\xi=0}^{2j}\binom{1+2j-\xi}{1}\frac{(i\theta)^\xi}{\xi!}\\ &\qquad\times\sum_{m,n=1}^{\infty}\frac{x^my^n}{m^sn^t(m+n)^{u+2+2j-\xi}(m+2n)^v}=0\quad(\theta\in[-\pi,\pi]).\end{aligned} \tag{9.85}$$

For simplicity, we denote the sum of the third and the fourth terms on the left-hand side of (9.85) by

$$-2\sum_{j=0,1}\phi(2-2j)\sum_{\xi=0}^{2j}\frac{(i\theta)^{\xi}}{\xi!}$$
$$\times\left[\sum_{m,n=1}^{\infty}\left\{(-1)^{m+n}\mathcal{D}_1(m,n;2j-\xi)e^{i(m+n)\theta}+\mathcal{D}_2(m,n;2j-\xi)\right\}x^m y^n\right],$$

where $\mathcal{D}_j(m,n;\nu)\in\mathbb{R}$ $(j=1,2)$. Since (9.85) holds for $y\in\mathbb{C}$ with $|y|=1$, we replace y by $-ye^{-i\theta}$ with $y\in\mathbb{C}$ $(|y|=1)$. Then we have

$$\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(l+m)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v (l+m+n)^2} \tag{9.86}$$
$$+\sum_{\substack{l,m,n=1\\ l\neq m+n,\ l\neq m}}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(-l+m)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v (-l+m+n)^2}$$
$$-2\sum_{j=0,1}\phi(2-2j)\sum_{\xi=0}^{2j}\sum_{m,n=1}^{\infty}\Big\{(-1)^m\mathcal{D}_1(m,n;2j-\xi)e^{im\theta}$$
$$+(-1)^n\mathcal{D}_2(m,n;2j-\xi)e^{-in\theta}\Big\}x^m y^n\frac{(i\theta)^{\xi}}{\xi!}$$
$$=-\sum_{m,n=1}^{\infty}\frac{x^m y^n}{m^{s+2}n^{t+2}(m+n)^u(m+2n)^v}\quad(\theta\in[-\pi,\pi]).$$

Therefore, applying Lemma 6.2 with $d=2$, we have

$$\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(l+m)\theta}}{l^2 m^s n^t (l+m)^2 (m+n)^u (m+2n)^v (l+m+n)^2} \tag{9.87}$$
$$=-\sum_{\substack{l,m,n=1\\ l\neq m+n,\ l\neq m}}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(-l+m)\theta}}{l^2 m^s n^t (-l+m)^2 (m+n)^u (m+2n)^v (-l+m+n)^2}$$
$$+2\sum_{j=0,1}\phi(2-2j)\sum_{\xi=0}^{2j}\sum_{m,n=1}^{\infty}\Big\{(-1)^m\mathcal{D}'_1(m,n;2j-\xi)e^{im\theta}$$
$$+(-1)^n\mathcal{D}'_2(m,n;2j-\xi)e^{-in\theta}+\mathcal{D}'_3(m,n;2j-\xi)\Big\}x^m y^n\frac{(i\theta)^{\xi}}{\xi!}$$

for $\theta\in[-\pi,\pi]$ with some $\mathcal{D}'_j(m,n;\nu)\in\mathbb{R}$ $(j=1,2,3)$.

Now we repeat this procedure. Namely, replace y by $ye^{2i\theta}$ and apply Lemma 6.2 with $d=2$. Furthermore, replace x by $-xe^{i\theta}$ and apply Lemma

6.2 with $d = 2$. Then we can obtain the equation

$$\sum_{l,m,n=1}^{\infty} \frac{(-1)^l x^m y^n e^{i(l+2m+2n)\theta}}{l^2 m^s n^t (l+m)^2 (m+n)^u (m+2n)^v (l+m+n)^2 (l+m+2n)^2 (l+2m+2n)^2} \tag{9.88}$$

$$\begin{aligned}
&= - \sum_{\substack{l,m,n=1 \\ l\neq m,\ l\neq m+n \\ l\neq m+2n,\ l\neq 2m+2n}}^{\infty} \frac{1}{l^2 m^s n^t (-l+m)^2 (m+n)^u (m+2n)^v (-l+m+n)^2} \\
&\qquad\qquad \times \frac{(-1)^l x^m y^n e^{i(-l+2m+2n)\theta}}{(-l+m+2n)^2 (-l+2m+2n)^2} \\
&+ 2 \sum_{j=0,1} \phi(2-2j) \sum_{\xi=0}^{2j} \sum_{m,n=1}^{\infty} \Big\{ \widetilde{\mathcal{D}}_1(m,n;2j-\xi) e^{i(2m+2n)\theta} \\
&\quad + (-1)^{m+n} \widetilde{\mathcal{D}}_2(m,n;2j-\xi) e^{i(m+n)\theta} + (-1)^m \widetilde{\mathcal{D}}_3(m,n;2j-\xi) e^{i(m+2n)\theta} \\
&\quad + (-1)^m \widetilde{\mathcal{D}}_4(m,n;2j-\xi) e^{im\theta} + \widetilde{\mathcal{D}}_5(m,n;2j-\xi) \Big\} x^m y^n \frac{(i\theta)^\xi}{\xi!}
\end{aligned}$$

for $\theta \in [-\pi, \pi]$ with some $\widetilde{\mathcal{D}}_j(m,n;\nu) \in \mathbb{R}$ $(j = 1, 2, \ldots, 5)$. Put $\theta = \pi$ and $(x, y) = (1, 1)$, and consider the real part. Then we have

$$\sum_{l,m,n=1}^{\infty} \frac{1}{l^2 m^s n^t (l+m)^2 (m+n)^u (m+2n)^v (l+m+n)^2 (l+m+2n)^2 (l+2m+2n)^2} \tag{9.89}$$

$$\begin{aligned}
&+ \sum_{\substack{l,m,n=1 \\ l\neq m,\ l\neq m+n \\ l\neq m+2n,\ l\neq 2m+2n}}^{\infty} \frac{1}{l^2 m^s n^t (-l+m)^2 (m+n)^u (m+2n)^v (-l+m+n)^2} \\
&\qquad\qquad \times \frac{1}{(-l+m+2n)^2 (-l+2m+2n)^2} \\
&- 2 \sum_{j=0,1} \phi(2-2j) \sum_{\tau=0}^{j} \sum_{m,n=1}^{\infty} \Big\{ \widetilde{\mathcal{D}}_1(m,n;2j-2\tau) + \widetilde{\mathcal{D}}_2(m,n;2j-2\tau) \\
&\quad + \widetilde{\mathcal{D}}_3(m,n;2j-2\tau) + \widetilde{\mathcal{D}}_4(m,n;2j-2\tau) + \widetilde{\mathcal{D}}_5(m,n;2j-2\tau) \Big\} \\
&\quad \times \frac{(-1)^\tau \pi^{2\tau}}{(2\tau)!} = 0.
\end{aligned}$$

By (9.72), we see that the first term on the left-hand side of (9.89) coincides with

$$\zeta_3(2, s, t, 2, u, v, 2, 2, 2; C_3).$$

For the second term on the left-hand side of (9.89), change the running indices of summation corresponding to the conditions $l \neq m$, $l \neq m+n$, $l \neq m+2n$, $l \neq 2m+2n$. Then we can see that the second term on the left-hand side of (9.89) coincides with

$$\begin{aligned}&\zeta_3(2, s, t, 2, u, v, 2, 2, 2; C_3) + 2\zeta_3(2, 2, t, s, 2, 2, u, v, 2; C_3)\\&\quad + 2\zeta_3(s, 2, 2, 2, t, 2, u, 2, v; C_3).\end{aligned}$$

Furthermore, using Lemma 6.1, we can rewrite the third term on the left-hand side of (9.89) as

$$\begin{aligned}&-2\sum_{\xi=0,1}\zeta(2\xi)\sum_{m,n=1}^{\infty}\Big\{\widetilde{\mathcal{D}}_1(m, n; 2-2\xi) + \widetilde{\mathcal{D}}_2(m, n; 2-2\xi)\\&\quad + \widetilde{\mathcal{D}}_3(m, n; 2-2\xi) + \widetilde{\mathcal{D}}_4(m, n; 2-2\xi) + \widetilde{\mathcal{D}}_5(m, n; 2-2\xi)\Big\}.\end{aligned}$$

We can concretely calculate the value $\widetilde{\mathcal{D}}_j(m, n; \nu)$ in terms of $\zeta_2(\mathbf{s}; C_2)$ and $\zeta(s)$. Combining these results, we obtain the assertion. □

Example 9.1. Putting $(s, t, u, v) = (2, 2, 2, 2)$, we obtain

$$\begin{aligned}
&3\,\zeta_3(2, 2, 2, 2, 2, 2, 2, 2, 2; C_3)\\
&= \zeta_2(5, 5, 4, 2; C_2)\zeta(2) - \frac{3}{2}\zeta_2(5, 5, 6, 2; C_2) + \frac{1}{2}\zeta_2(6, 4, 4, 2; C_2)\zeta(2) - \frac{3}{4}\zeta_2(6, 4, 6, 2; C_2)\\
&+ \zeta_2(6, 5, 3, 2; C_2)\zeta(2) - \frac{3}{2}\zeta_2(6, 5, 5, 2; C_2) - \frac{3}{2}\zeta_2(6, 8, 2, 2; C_2) + \frac{3}{4}\zeta_2(7, 4, 3, 2; C_2)\zeta(2)\\
&- \frac{9}{8}\zeta_2(7, 4, 5, 2; C_2) - \frac{11}{16}\zeta_2(7, 5, 4, 2; C_2) - \frac{23}{32}\zeta_2(8, 4, 4, 2; C_2) - \zeta_2(8, 6, 2, 2; C_2)\\
&+ \zeta_2(4, 5, 4, 3; C_2)\zeta(2) - \frac{3}{2}\zeta_2(4, 5, 6, 3; C_2) + \frac{1}{2}\zeta_2(5, 5, 3, 3; C_2)\zeta(2) - \frac{3}{4}\zeta_2(5, 5, 5, 3; C_2)\\
&+ \zeta_2(6, 5, 2, 3; C_2)\zeta(2) - \frac{3}{16}\zeta_2(6, 5, 4, 3; C_2) - \frac{3}{4}\zeta_2(7, 4, 2, 3; C_2)\zeta(2) + \zeta_2(2, 5, 5, 4; C_2)\zeta(2)\\
&+ \frac{3}{2}\zeta_2(2, 6, 4, 4; C_2)\zeta(2) - \frac{3}{8}\zeta_2(2, 6, 6, 4; C_2) - \frac{7}{8}\zeta_2(2, 7, 5, 4; C_2) - \frac{15}{16}\zeta_2(2, 8, 4, 4; C_2)\\
&+ \frac{1}{2}\zeta_2(4, 4, 4, 4; C_2)\zeta(2) - \frac{3}{4}\zeta_2(4, 4, 6, 4; C_2) - \frac{1}{2}\zeta_2(4, 5, 3, 4; C_2)\zeta(2) + \frac{3}{4}\zeta_2(4, 5, 5, 4; C_2)\\
&+ \frac{1}{4}\zeta_2(5, 4, 3, 4; C_2)\zeta(2) - \frac{3}{8}\zeta_2(5, 4, 5, 4; C_2) - \frac{3}{2}\zeta_2(5, 5, 2, 4; C_2)\zeta(2) + \frac{1}{4}\zeta_2(5, 5, 4, 4; C_2)
\end{aligned}$$

$$+\frac{1}{2}\zeta_2(6,4,2,4;C_2)\zeta(2)-\frac{3}{32}\zeta_2(6,4,4,4;C_2)+\frac{11}{16}\zeta_2(7,5,2,4;C_2)-\frac{23}{32}\zeta_2(8,4,2,4;C_2)$$
$$-2\zeta_2(2,4,5,5;C_2)\zeta(2)-\zeta_2(2,5,4,5;C_2)\zeta(2)-\frac{1}{2}\zeta_2(2,6,5,5;C_2)-\frac{7}{8}\zeta_2(2,7,4,5;C_2)$$
$$-\frac{1}{2}\zeta_2(4,4,3,5;C_2)\zeta(2)+\frac{3}{4}\zeta_2(4,4,5,5;C_2)+\frac{3}{2}\zeta_2(4,5,2,5;C_2)\zeta(2)-\frac{1}{4}\zeta_2(4,5,4,5;C_2)$$
$$-\frac{1}{4}\zeta_2(5,4,2,5;C_2)\zeta(2)+\frac{1}{4}\zeta_2(5,4,4,5;C_2)-\frac{27}{16}\zeta_2(6,5,2,5;C_2)+\frac{9}{8}\zeta_2(7,4,2,5;C_2)$$
$$-\frac{3}{2}\zeta_2(2,4,6,6;C_2)+4\zeta_2(2,5,3,6;C_2)\zeta(2)-\zeta_2(2,5,5,6;C_2)-\frac{3}{8}\zeta_2(2,6,4,6;C_2)$$
$$-2\zeta_2(3,5,2,6;C_2)\zeta(2)+\zeta_2(4,4,2,6;C_2)\zeta(2)-\frac{3}{8}\zeta_2(4,4,4,6;C_2)+\frac{19}{8}\zeta_2(5,5,2,6;C_2)$$
$$-\frac{27}{32}\zeta_2(6,4,2,6;C_2)+4\zeta_2(2,4,3,7;C_2)\zeta(2)+2\zeta_2(2,4,5,7;C_2)+3\zeta_2(2,5,4,7;C_2)$$
$$-2\zeta_2(3,4,2,7;C_2)\zeta(2)-\frac{9}{4}\zeta_2(4,5,2,7;C_2)+\frac{1}{2}\zeta_2(5,4,2,7;C_2)-6\zeta_2(2,5,3,8;C_2)$$
$$+3\zeta_2(3,5,2,8;C_2)-\frac{3}{2}\zeta_2(4,4,2,8;C_2)-6\zeta_2(2,4,3,9;C_2)+3\zeta_2(3,4,2,9;C_2).$$

The authors also checked this equation numerically by using definitions (8.64) and (9.72).

Remark 9.2. From these considerations, we can see that our method may be applied to much wider class of multiple zeta-functions. As another example, we will consider the zeta-function $\zeta_2(\mathbf{s};G_2)$ associated with the exceptional Lie algebra of type G_2 and will give certain functional relations including explicit forms of Witten's volume formulas of type G_2 in a forthcoming paper [20].

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