SUM FORMULAS FOR DOUBLE POLYLOGARITHMS OF HURWITZ TYPE AND SOME APPLICATIONS

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1. Introduction

In this note, we introduce sum formulas for double polylogarithms of Hurwitz type including well-known sum formulas for double zeta values. Also we give certain weighted sum formulas for double polylogarithms, further for double L-values. Moreover we give certain sum formulas for partial double zeta values attached to congruent conditions.

For the details in this note, see the joint-papers: Matsumoto-Tsumura [8] and Essouabri-Matsumoto-Tsumura [3].

2. Various sum formulas for double zeta values

We recall various sum formulas for the double zeta value $\zeta(p,q)=\sum_{m>n\geq 1}\frac{1}{m^p n^q}$:

Sum formula for double zeta values For $l\geq 3$,

$$\sum_{j=2}^{l-1}\zeta(j,l-j)=\zeta(l).$$

Gangl-Kaneko-Zagier [4] For $N\geq 2$,

$$\sum_{\nu=1}^{N-1}\zeta(2\nu,2N-2\nu)=\frac{3}{4}\zeta(2N);\quad \sum_{\nu=1}^{N-1}\zeta(2\nu+1,2N-2\nu-1)=\frac{1}{4}\zeta(2N).$$

Ohno-Zudilin [10] For $l\geq 3$,

$$\sum_{\nu=2}^{l-1}2^{\nu}\zeta(\nu,l-\nu)=(l+1)\zeta(l).$$

This was extended to the relevant result for multiple zeta values by **Guo and Xie** [5].

Nakamura [9] For $N\geq 2$ and $M\geq 4$,

$$\sum_{\nu=1}^{N-1}\left(4^{\nu}+4^{N-\nu}\right)\zeta(2\nu,2N-2\nu)=\left(N+\frac{4}{3}+\frac{2}{3}4^{N-1}\right)\zeta(2N);$$
$$\sum_{\nu=2}^{M-2}(2\nu-1)(2M-2\nu-1)\zeta(2\nu,2M-2\nu)=\frac{3}{4}(M-3)\zeta(2M).$$

Based on these facts, we aim to prove sum formulas for double polylogarithms of Hurwitz type as follows.

Theorem 2.1 ([8]). *For $k \in \mathbb{N}$, $x \in \mathbb{C}$ with $|x| \le 1$ and $b \in (0,1]$,*

$$\begin{aligned}
&\cos(b\pi)\Bigg\{\sum_{\substack{m\ge1\\n\ge0}}\frac{x^n}{m(m+n+b)^{k+1}}-\sum_{\substack{m\ge1\\n\ge0}}\frac{x^{m+n}}{m(m+n+b)^{k+1}}+\sum_{\nu=2}^{k+1}\sum_{\substack{m\ge1\\n\ge0}}\frac{x^n}{(n+b)^{k+2-\nu}(m+n+b)^{\nu}}\\
&\quad-\sum_{\substack{m\ge1\\n\ge0}}\frac{x^{m+n}}{(n+b)(m+n+b)^{k+1}}+\sum_{\substack{m>b\\n\ge0}}\frac{x^n}{(m-b)(m+n)^{k+1}}+\sum_{\nu=2}^{k+1}\sum_{\substack{m>b\\n\ge0}}\frac{x^n}{(n+b)^{k+2-\nu}(m+n)^{\nu}}\Bigg\}\\
&-\frac{\sin(b\pi)}{\pi}\Bigg\{\sum_{\substack{m\ge1\\n\ge0}}\frac{x^n}{m(m+n+b)^{k+2}}-\sum_{\substack{m\ge1\\n\ge0}}\frac{x^{m+n}}{m(m+n+b)^{k+2}}+\sum_{\nu=1}^{k}\sum_{\substack{m\ge1\\n\ge0}}\frac{x^n}{(n+b)^{\nu}(m+n+b)^{k+3-\nu}}\\
&\quad-\sum_{\substack{m\ge1\\n\ge0}}\frac{x^{m+n}}{(n+b)(m+n+b)^{k+2}}-\sum_{\substack{m\ge1\\n\ge0}}\frac{x^{m+n}}{(n+b)^2(m+n+b)^{k+1}}-\sum_{\substack{m>b\\n\ge0}}\frac{x^n}{(m-b)^2(m+n)^{k+1}}\\
&\quad-k\sum_{\substack{m>b\\n\ge0}}\frac{x^n}{(m-b)(m+n)^{k+2}}-\sum_{\nu=1}^{k}\nu\sum_{\substack{m>b\\n\ge0}}\frac{x^n}{(n+b)^{k+1-\nu}(m+n)^{\nu+2}}\Bigg\}\\
&=\pi\sin(b\pi)\sum_{n=0}^{\infty}\frac{x^n}{(n+b)^{k+1}}+2\cos(b\pi)\sum_{n=0}^{\infty}\frac{x^n}{(n+b)^{k+2}}-\frac{2\sin(b\pi)}{\pi}\sum_{n=0}^{\infty}\frac{x^n}{(n+b)^{k+3}}. \qquad (\star)
\end{aligned}$$

Note that the case $(b,x)=(1,1)$ implies the ordinary sum formula.

Corollary 2.2 ([3]). *For $k \in \mathbb{N}$ and $x \in \mathbb{C}$ with $|x| \le 1$,*

$$\sum_{\substack{m\ge1\\n\ge1}}\frac{x^n}{m(m+n)^{k+1}}-\sum_{\substack{m\ge1\\n\ge1}}\frac{x^{m+n}}{m(m+n)^{k+1}}+\sum_{\nu=2}^{k+1}\sum_{\substack{m\ge1\\n\ge1}}\frac{x^n}{n^{k+2-\nu}(m+n)^{\nu}}=\mathrm{Li}(k+2;x),$$

where $\mathrm{Li}(s;x)=\sum_{n\ge1}x^n n^{-s}$. Furthermore, for a primitive Dirichlet character χ,

$$\sum_{\nu=2}^{k+1}L_{ш}(\nu,k+2-\nu;\chi_0,\chi)+L_{ш}(k+1,1;\chi,\chi_0)-L_{*}(k+1,1;\chi,\chi_0)=L(k+2;\chi),$$

where χ_0 is the trivial character, $L(s,\chi):=\sum_{m\ge1}\chi(m)m^{-s}$ is the Dirichlet L-series and

$$L_{*}(s_1,s_2;\psi_1,\psi_2):=\sum_{m_1>m_2\ge1}\frac{\psi_1(m_1)\psi_2(m_2)}{m_1^{s_1}m_2^{s_2}}=\sum_{\substack{m\ge1\\n\ge1}}\frac{\psi_2(m)\psi_1(m+n)}{m^{s_2}(m+n)^{s_1}},$$

$$L_{ш}(s_1,s_2;\psi_1,\psi_2):=\sum_{m_1>m_2\ge1}\frac{\psi_1(m_1-m_2)\psi_2(m_2)}{m_1^{s_1}m_2^{s_2}}=\sum_{\substack{m\ge1\\n\ge1}}\frac{\psi_2(m)\psi_1(n)}{m^{s_2}(m+n)^{s_1}}$$

are double L-series first studied in Arakawa-Kaneko [2].

Theorem 1 holds as a *functional relation* among holomorphic functions for $b \in D_\varepsilon(1)$, where we fix a sufficiently small $\varepsilon > 0$ and let

$$D_\varepsilon(1) = \{b \in \mathbb{C} \mid |b-1| < \varepsilon\}.$$

By differentiating Theorem 1 in b and $b \to 1$, we obtain

Corollary 2.3. *For $k \in \mathbb{N}$ and $x \in \mathbb{C}$ with $|x| \le 1$,*

$$\sum_{\nu=2}^{k+1} \nu \sum_{\substack{m\ge1\\ n\ge1}} \frac{x^n}{n^{k+2-\nu}(m+n)^{\nu+1}} + (k+1) \sum_{\substack{m\ge1\\ n\ge1}} \frac{x^n}{m(m+n)^{k+2}}$$
$$+ \sum_{\substack{m\ge1\\ n\ge1}} \frac{x^n}{m^2(m+n)^{k+1}} - \sum_{\substack{m\ge1\\ n\ge1}} \frac{x^{m+n}}{m^2(m+n)^{k+1}} = \sum_{n=1}^{\infty} \frac{x^n}{n^{k+3}}.$$

Corollary 2.4. *For $k \in \mathbb{N}$ and a primitive Dirichlet character χ,*

$$\sum_{\nu=2}^{k+1} \nu\, L_{ш}(\nu+1, k+2-\nu; \chi_0, \chi) + (k+1) L_{ш}(k+2, 1; \chi, \chi_0)$$
$$+ L_{ш}(k+1, 2; \chi, \chi_0) - L_*(k+1, 2; \chi, \chi_0) = L(k+3; \chi).$$

In particular,

$$\sum_{\nu=2}^{k} \nu\, \zeta(\nu+1, k+2-\nu) + 2(k+1)\zeta(k+2, 1) = \zeta(k+3). \qquad (*)$$

Note that $(*)$ can also be derived from the result of Arakawa-Kaneko multiple zeta-function in [1] as follows.

Arakawa-Kaneko multiple zeta-function

$$\xi(k_1, \ldots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \cdot \mathrm{Li}_{k_1,\ldots,k_r}(1 - e^{-t}) dt,$$

where

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{0<m_1<\cdots<m_r} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \qquad (k_1, \ldots, k_r \in \mathbb{N}).$$

Indeed Arakawa-Kaneko showed that

$$\xi(2; k+1) = -\zeta(k+1, 2) - (k+1)\zeta(k+2, 1) + \zeta(2)\zeta(k+1);$$
$$\xi(2; k+1) = \sum_{\nu=1}^{k+1} \nu \zeta(\nu+1, k+2-\nu).$$

Combining these results, we obtain $(*)$.

Remark 2.5. Arakawa-Kaneko essentially gave general form of relation formulas including $(*)$ for '*multiple zeta values*', by using the analytic property of their ξ-function and the duality of MZVs.

By differentiating Theorem 1 in b one more time, we can obtain

$$\sum_{\nu=3}^{k}\frac{\nu(\nu-1)}{2}\zeta(\nu+1,k+3-\nu)+\frac{(k+1)(k+2)}{2}\zeta(k+2,2)$$
$$+(k+1)(k+2)\zeta(k+3,1)=\zeta(k+4)\qquad (k\geq 3),$$

which is given from their result on $\xi(1,2;k+1)$.

3. Sketch of the proof of Theorem 2.1

First we prepare the following functional relation.

Theorem 3.1. *For $b\in\mathbb{R}$ with $0<b\leq 1$, $x\in\mathbb{C}$ with $|x|\leq 1$ and $s\in\mathbb{C}$ with $\Re s\geq 1$,*

$$\cos(b\pi)\Bigg\{\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{m(n+b)^s(m+n+b)}-\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^{m+n}}{m(n+b)(m+n+b)^s}$$
$$+\sum_{\substack{m>b\\ n\geq 0}}\frac{x^n}{(m-b)(n+b)^s(m+n)}\Bigg\}$$
$$-\frac{\sin(b\pi)}{\pi}\Bigg\{\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{m(n+b)^s(m+n+b)^2}-\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^{m+n}}{m(n+b)^2(m+n+b)^s}$$
$$+\sum_{\substack{m>b\\ n\geq 0}}\frac{x^n}{(m-b)^2(n+b)^s(m+n)}\Bigg\}$$
$$=\pi\sin(b\pi)\sum_{n\geq 0}\frac{x^n}{(n+b)^{s+1}}-2\cos(b\pi)\sum_{n\geq 0}\frac{x^n}{(n+b)^{s+2}}+\frac{\sin(b\pi)}{\pi}\sum_{n\geq 0}\frac{x^n}{(n+b)^{s+3}}. \tag{3.1}$$

Proof. Assume $\Re s>1$. By

$$\lim_{M\to\infty}\sum_{m=1}^{M}\frac{(-1)^m\sin(m\theta)}{m}=-\frac{\theta}{2}\qquad (\theta\in(-\pi,\pi)),$$

we have

$$2i\left(\lim_{M\to\infty}\sum_{m=1}^{M}\frac{(-1)^m\sin(m\theta)}{m}+\frac{\theta}{2}\right)\sum_{n\geq 0}\frac{(-1)^n x^n e^{i(n+b)\theta}}{(n+b)^s}=0.$$

Hence we see that

$$\lim_{M\to\infty}\sum_{m=1}^{M}\sum_{n\geq 0}\frac{(-1)^{m+n}x^n e^{i(m+n+b)\theta}}{m(n+b)^s}-\lim_{M\to\infty}\sum_{m=1}^{M}\sum_{\substack{n\geq 0\\ m\neq n+b}}\frac{(-1)^{m+n}x^n e^{i(-m+n+b)\theta}}{m(n+b)^s}$$
$$+i\theta\sum_{n\geq 0}\frac{(-1)^n x^n e^{in\theta}}{(n+b)^s}=\begin{cases}-\sum_{n\geq 0}\frac{x^n}{(n+1)^{s+1}} & (b=1)\\ 0 & (b\neq 1),\end{cases} \tag{$**$}$$

where the change of order of summation is possible (see [6]). Denote the left-hand side by $J(\theta)$ and the right-hand side by C_0, respectively.

Multiply by $(i/2\pi)\theta$ on the both sides and consider the integration

$$\frac{i}{2\pi}\int_{-\pi}^{\pi}\theta J(\theta)d\theta = \frac{iC_0}{2\pi}\int_{-\pi}^{\pi}\theta\, d\theta = 0.$$

This gives that

$$\begin{aligned}
&\cos(b\pi)\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{m(n+b)^s(m+n+b)} - \frac{\sin(b\pi)}{\pi}\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{m(n+b)^s(m+n+b)^2}\\
&-\cos(b\pi)\sum_{\substack{m\geq 1\\ n\geq 0\\ m\neq n+b}}\frac{x^n}{m(n+b)^s(m+n+b)} + \frac{\sin(b\pi)}{\pi}\sum_{\substack{m\geq 1\\ n\geq 0\\ m\neq n+b}}\frac{x^n}{m(n+b)^s(m+n+b)^2}\\
&-\pi\sin(b\pi)\sum_{n\geq 0}\frac{x^n}{(n+b)^{s+1}} - 2\cos(b\pi)\sum_{n\geq 0}\frac{x^n}{(n+b)^{s+2}}\\
&+\frac{\sin(b\pi)}{\pi}\sum_{n\geq 0}\frac{x^n}{(n+b)^{s+3}} = 0,
\end{aligned}$$

which further converges absolutely for $\Re s \geq 1$. This completes the proof. $\square$

Sketch of the proof of Theorem 2.1. Letting $s = k \in \mathbb{N}$ in (3.1) and using repeatedly the partial fraction decomposition

$$\frac{1}{XY} = \left(\frac{1}{X}+\frac{1}{Y}\right)\frac{1}{X+Y}.$$

Then former three terms in the parentheses on the left-hand side of (3.1) is

$$\begin{aligned}
&\sum_{\nu=2}^{k+1}\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{(n+b)^{k+2-\nu}(m+n+b)^{\nu}} + \sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{m(m+n+b)^{k+1}}\\
&-\sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{(n+b)(m+n+b)^{k+1}} - \sum_{\substack{m\geq 1\\ n\geq 0}}\frac{x^n}{m(m+n+b)^{k+1}}\\
&+\sum_{\nu=2}^{k+1}\sum_{\substack{m>b\\ n\geq 0}}\frac{x^n}{(n+b)^{k+2-\nu}(m+n)^{\nu}} + \sum_{\substack{m>b\\ n\geq 0}}\frac{x^n}{(m-b)(m+n)^{k+1}}.
\end{aligned}\tag{3.2}$$

We can similarly decompose the other terms, so we obtain Theorem 2.1. $\square$

Fix $k \in \mathbb{N}$, $x \in \mathbb{C}$ with $|x| \geq 1$, assume $0 < b < 1$, and differentiate the both sides of $(\star)$ in Theorem 2.1 with respect to b.

It should be noted that on the left-hand side of $(\star)$ in Theorem 2.1,

$$\sum_{\substack{m>b\\ n\geq 0}}\frac{x^n}{(m-b)(m+n)^{k+2}} = \frac{1}{x(1-b)}Li(k+2;x) + \sum_{\substack{m\geq 2\\ n\geq 0}}\frac{x^n}{(m-b)(m+n)^{k+2}},$$

$$\sum_{\substack{m>b\\ n\geq 0}} \frac{x^n}{(m-b)^2(m+n)^{k+1}} = \frac{1}{x(1-b)^2}Li(k+1;x) + \sum_{\substack{m\geq 2\\ n\geq 0}} \frac{x^n}{(m-b)^2(m+n)^{k+1}},$$

$$\sum_{\substack{m>b\\ n\geq 0}} \frac{x^n}{(m-b)(m+n)^{k+2}} = \frac{1}{x(1-b)}Li(k+2;x) + \sum_{\substack{m\geq 2\\ n\geq 0}} \frac{x^n}{(m-b)(m+n)^{k+2}}$$

are not continuous at $b=1$. Hence we consider $g(b) = g_k(b;x)$ defined by

$$g(b) = \frac{\cos(b\pi)}{x(1-b)}Li(k+1;x) + \frac{\sin(b\pi)}{\pi x(1-b)^2}Li(k+1;x) + \frac{k\sin(b\pi)}{\pi x(1-b)}Li(k+2;x).$$

Let $k \in \mathbb{N}$ and $x \in \mathbb{C}$ with $|x| \leq 1$. Then we can easily check that $g(b) = g_k(b;x)$ is holomorphic for $b \in D_\varepsilon(1) = \{b \in \mathbb{C} \mid |b-1| < \varepsilon\}$ and satisfies

$$g(1) = \frac{k}{x}Li(k+2;x),\ g'(1) = -\frac{\pi^2}{3x}Li(k+1;x),\ g''(1) = -\frac{k\pi^2}{3x}Li(k+2;x).$$

Namely $(\star)$ holds as a functional relation among holomorphic functions for $b \in D_\varepsilon(1)$.

Therefore we can differentiate the both sides of $(\star)$ in Theorem 2.1 with respect to b, and can prove Corollaries 2.3 and 2.4.

4. Sum formulas attached to congruent conditions

Theorem 4.1. *Let N be odd positive. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$ with $|x| \leq 1$,*

$$\begin{aligned}
&\sum_{\nu=1}^{k}\Bigg\{\sum_{\substack{m,n\geq 1\\ m\equiv n\ (mod\ N)}} \frac{x^n}{n^\nu(m+n)^{k+2-\nu}}\Bigg\} + \sum_{\substack{m,n\geq 1\\ m\equiv n\ (mod\ N)}} \frac{x^n}{m(m+n)^{k+1}}\\
&\quad + \sum_{\nu=1}^{k}\Bigg\{\sum_{\substack{m,n\geq 1\\ m\equiv -2n\ (mod\ N)}} \frac{x^n}{n^\nu(m+n)^{k+2-\nu}}\Bigg\} + \sum_{\substack{m,n\geq 1\\ m\equiv -2n\ (mod\ N)}} \frac{x^n}{m(m+n)^{k+1}}\\
&\quad - \sum_{\substack{m,n\geq 1\\ m\equiv -2n\ (mod\ N)}} \frac{x^{m+n}}{m(m+n)^{k+1}} - \sum_{\substack{m,n\geq 1\\ m\equiv -2n\ (mod\ N)}} \frac{x^{m+n}}{n(m+n)^{k+1}}\\
&\quad = \frac{2}{N^{k+2}}Li(k+2;x^N) + \frac{\pi}{N}\sum_{\substack{m=1\\ N\nmid m}}^{\infty} \frac{x^m}{\sin(2m\pi/N)m^{k+1}}.
\end{aligned}$$

Note that the case $N=1$ and $x=1$ implies the ordinary sum formulas.

In the case $N=3$ and $x=1$, we see that $m \equiv -2n \pmod 3$ implies $m \equiv n \pmod 3$. Let $\chi_3(m) = (2/\sqrt{3})\sin(2m\pi/3)$ be the quadratic character of conductor 3. Then we have the following.

Corollary 4.2. *For $k \in \mathbb{N}$,*

$$\sum_{\nu=1}^{k}\Bigg\{\sum_{\substack{m,n\geq 1\\ m\equiv n\ (mod\ 3)}} \frac{1}{n^\nu(m+n)^{k+2-\nu}}\Bigg\} = \frac{1}{3^{k+2}}\zeta(k+2) + \frac{\pi}{3\sqrt{3}}L(k+1;\chi_3).$$

In particular when $k=1$,

$$\sum_{\substack{m,n\geq 1\\ m\equiv n\ (mod\ 3)}} \frac{1}{n(m+n)^2} = \frac{1}{27}\zeta(3) + \frac{\pi}{3\sqrt{3}}L(2;\chi_3).$$

Sketch of the proof of Theorem 4.1. Similarly to Theorem 3.1, we obtain

$$\begin{aligned}&\sum_{\substack{m,n\geq 1\\ m\equiv n\ (\mathrm{mod}\ N)}} \frac{x^n}{mn^s(m+n)} + \sum_{\substack{m,n\geq 1\\ m\equiv -2n\ (\mathrm{mod}\ N)}} \frac{x^n}{mn^s(m+n)} - \sum_{\substack{m,n\geq 1\\ m\equiv -2n\ (\mathrm{mod}\ N)}} \frac{x^{m+n}}{mn(m+n)^s}\\ &= \frac{2}{N^{s+2}}Li(s+2;x^N) + \frac{\pi}{N}\sum_{\substack{m=1\\ N\nmid m}}^{\infty} \frac{x^m}{\sin(2m\pi/N)m^{s+1}}\end{aligned} \tag{$\star\star$}$$

for $s\in\mathbb{C}$ with $\Re s\geq 1$, by letting $b=1$, $\eta=e^{2\pi i/N}$ for an odd positive integer N, $x=\eta^{-2a}$ in $(**)$ and using

$$\sum_{a=0}^{N-1} a\rho^{-an} = \begin{cases} \frac{N(N-1)}{2} & (N\mid n)\\ \frac{N}{\rho^{-n}-1} & (N\nmid n).\end{cases}$$

where $\rho=\eta,\eta^2$. Then, setting $s=k\in\mathbb{N}$ in $(\star\star)$ and using the partial fraction decomposition, we obtain the proof. □

5. The case of a general depth

First we consider the triple L-values.

Theorem 5.1 ([3]). *For $K\in\mathbb{N}$ with $K>3$ and a Dirichlet character χ,*

$$\begin{aligned}&\sum_{\substack{k_1,k_2\geq 1,\ k_3\geq 2\\ k_1+k_2+k_3=K}} \left\{\sum_{m_1,m_2,m_3\in\mathbb{N}} \frac{\chi(m_1)}{m_1^{k_1}(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}}\right\}\\ &+\sum_{\substack{k_2\geq 1,\ k_3\geq 2\\ k_2+k_3=K-1}} \left\{\sum_{m_1,m_2,m_3\in\mathbb{N}} \frac{\chi(m_2)-\chi(m_1+m_2)}{m_1(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}}\right\}\\ &+\sum_{m_1,m_2,m_3\in\mathbb{N}} \frac{\chi(m_3)-\chi(m_1+m_3)-\chi(m_2+m_3)+\chi(m_1+m_2+m_3)}{m_1(m_1+m_2)(m_1+m_2+m_3)^{K-2}}\\ &\quad=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^K}.\end{aligned}$$

Note that when $\chi=\chi_0$, we obtain the ordinary *sum formula* for triple zeta values. The method of this proof is essentially the same as in the case of double series, though the calculation is much harder.

From the double and triple cases, we expect the following result.

Conjecture 5.2 ([3]). *For $r \in \mathbb{N}$ and $K \in \mathbb{N}$ with $K > r$,*

$$\sum_{\substack{k_1,\ldots,k_{r-1}\geq 1,\ k_r\geq 2\\ k_1+\cdots+k_r=K}} \left\{ \sum_{m_1,m_2,\ldots,m_r\in\mathbb{N}} \frac{\chi(m_1)}{m_1^{k_1}(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}\cdots(\sum_{j=1}^r m_j)^{k_r}} \right\}$$
$$+ \sum_{\substack{k_2,\ldots,k_{r-1}\geq 1,\ k_r\geq 2\\ k_2+\cdots+k_r=K-1}} \left\{ \sum_{m_1,m_2,\ldots,m_r\in\mathbb{N}} \frac{\chi(m_2)-\chi(m_1+m_2)}{m_1(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}\cdots(\sum_{j=1}^r m_j)^{k_r}} \right\}$$
$$+ \sum_{\substack{k_3,\ldots,k_{r-1}\geq 1,\ k_r\geq 2\\ k_3+\cdots+k_r=K-2}} \left\{ \sum_{m_1,m_2,\ldots,m_r\in\mathbb{N}} \frac{\chi(m_3)-\chi(m_1+m_3)-\chi(m_2+m_3)+\chi(m_1+m_2+m_3)}{m_1(m_1+m_2)(m_1+m_2+m_3)^{k_3}\cdots(\sum_{j=1}^r m_j)^{k_r}} \right\}$$
$$+\cdots$$
$$+ \sum_{m_1,m_2,\ldots,m_r\in\mathbb{N}} \frac{\chi(m_r)-\sum_{j<r}\chi(m_j+m_r)+\sum_{j_1<j_2<r}\chi(m_{j_1}+m_{j_2}+m_r)-\cdots}{m_1(m_1+m_2)(m_1+m_2+m_3)\cdots(\sum_{j=1}^{r-1} m_j)(\sum_{j=1}^r m_j)^{K-r+1}}$$
$$= \sum_{m=1}^\infty \frac{\chi(m)}{m^K}.$$

For example, the case $K = r+1$ (hence the only possible choice is $(k_1, k_2, \ldots, k_{r-1}, k_r) = (1, 1, \ldots, 1, 2)$) implies that

$$\sum_{m_1,m_2,\ldots,m_r\in\mathbb{N}} \frac{\sum_{j=1}^r \chi(m_j)-\sum_{j_1<j_2}\chi(m_{j_1}+m_{j_2})+\cdots+(-1)^{r-1}\chi(\sum_{j=1}^r m_j)}{m_1(m_1+m_2)(m_1+m_2+m_3)\cdots(\sum_{j=1}^{r-1} m_j)(\sum_{j=1}^r m_j)^2}$$
$$= \sum_{m=1}^\infty \frac{\chi(m)}{m^{r+1}}.$$

In particular when $\chi = \chi_0$, this coincides with the well-known formula

$$\zeta(2, 1, 1, \cdots, 1) = \zeta(r+1).$$

6. Conclusion remarks

At the end of this note, we list several remarks as follows:

(1) Can the result for double L-values in Corollary 2.4 be given from the property of Arakawa-Kaneko ξ-L-function?

(2) Are there certain connections between our results for multiple L-values and the cyclic sum formulas for multiple L-values? Indeed, can Conjecture 5.2 be given from the cyclic sum formulas for multiple L-values?

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